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VOLUME 2 - Mathematical Analysis

by

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MAY 1963

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STUDIES IN STRESS ANALYSIS OF VISCOELASTIC SOLIDS UNDER NON-STEADY TEMPERATURE GRAVITATIONAL AND INERTIAL LOADS,

by

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TABLE OF CONTENTS

LIST OF FIGURES

406~~128~~ 339

ABSTRACT

The object of the present work is the investigation into the problems of stress and deformation of linear viscoelastic solids under various environmental conditions, such as gravity, temperature and inertia forces.

Apart from the introduction, which deals with the state of the art to-date, the body of the report is divided in three chapters.

In the first chapter, the thermal stresses in viscoelastic solids with temperature dependent properties are investigated, temperature dependence being limited to the thermorheologically simple type. Here approximate methods are developed. Finally, the infinite cylinder under a transient radial temperature field is studied and two analytic solutions are given.

The second chapter is occupied with the evaluation of the effects of the acceleration terms on the stress distribution in incompressible viscoelastic solids, again under conditions of non-uniform transient temperature. Analytic solutions are given to the problems of the hollow sphere and cylinder.

The third chapter is devoted to gravitational effects. The problem of the horizontal slump of a viscoelastic hollow cylinder contained in a thin elastic shell and resting on a rigid horizontal plane is solved by the application of the Finite Fourier Transform and the Laplace Transform.

In appendix I the problem of an infinite slab and a solid sphere are solved numerically for the purpose of illustration and compared with exact solutions. The methods of the first chapter are also used

for numerical solution of the problem of cylinder in connection with digital computer programs. In Appendix II error estimates connected with one of the analytic methods are derived. In Appendix III the convergence of the first analytic method is established.

GENERAL INTRODUCTION

The method of analysis of the quasi-static boundary value problem for homogeneous isotropic linear viscoelastic materials under surface tractions, body forces and isothermal temperature conditions is now well established.

The formal solution of the above problem within the scope of small deformations and time independent boundaries can be effected by the application of Laplace transform thereby eliminating the time dependence and thus reducing the viscoelastic problem to an "associate" elastic one.

The introduction of a temperature field however, be it transient or steady, introduces new difficulties that are not easily surmountable.

The temperature effects are two-fold, i.e. (a) Thermal strains are set up, and (b) the mechanical properties being extremely sensitive to temperature variations - orders of 10° C are significant - ensure that a non-homogeneous transient temperature entails a non-homogeneous material with mechanical properties as functions of the space variables and time.

The mode of variation of the mechanical properties with temperature is in itself a problem which has only been partially solved.

In the present paper, these difficulties will be formally investigated and some methods for overcoming them will be given.

Review of the Isothermal Quasistatic Problem

Given a viscoelastic body B with surface S under surface tractions $T_i^{(s)}$ body forces f_i and surface displacements $u_i(s)$ which may possibly

depend on time, the object is to determine the stress and displacement distribution within the body.

The viscoelastic constitutive relations in Cartesian coordinates can always be put in the form*

$$s_{ij} = \int_0^t G_1(t-\tau)\frac{\partial e_{ij}}{\partial \tau}\,d\tau \qquad (1)$$

$$\sigma_{kk} = \int_0^t G_2(t-\tau)\frac{\partial \epsilon_{kk}}{\partial \tau}\,d\tau \qquad (2)$$

where

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \quad , \quad e_{ij} = \epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij} \qquad (3)$$

G_1 is the relaxation modulus in shear and G_2 is the relaxation modulus in dilatation.

The complete solution is obtained if the following relations are satisfied.

Equilibrium condition**

$$\sigma_{ij,j} + f_i = 0 \qquad (4)$$

Strain displacement relation

$$\epsilon_{ij} = \frac{1}{2}\left\{u_{i,j} + u_{j,i}\right\} \qquad (5)$$

---

* Repeated indices denote summation.

** A comma followed by an index j denote differentiation with respect to the j^{th} co-ordinate.

and the boundary conditions:

$$u_i = u_i^{(S_1)} \tag{6}$$

where $u_i^{(S_1)}$ are prescribed on the part S_1 of the boundary

$$T_i^{(S_2)} = \sigma_{ij} n_j \tag{7}$$

where $T_i^{(S_2)}$ are prescribed on the remaining part S_2 of the boundary S.

The usual approach is to remove time dependence by applying Laplace Transform.

Then Eq. (1) and (2) become

$$\bar{S}_{ij}(x_k, p) = p\,\bar{G}_1(p)\,\bar{e}_{ij}(x_k, p) \tag{8}$$

$$\bar{\sigma}_{ii}(x_k, p) = p\,\bar{G}_2(p)\,\epsilon_{ii}(x_k, p) \tag{9}$$

for an initially unstressed and unstrained state.

$$\bar{\sigma}_{ij,j} + \bar{f}_i = 0 \tag{10}$$

$$\bar{\epsilon}_{ij} = \frac{1}{2}\left\{\bar{u}_{i,j} + \bar{u}_{j,i}\right\} \tag{11}$$

$$\bar{u}_i = \bar{u}_i^{(S_1)} \qquad \text{on } S_1 \tag{12}$$

$$\bar{T}_i^{(S_2)} = \bar{\sigma}_{ij} n_j \qquad \text{on } S_2 \qquad (13)$$

Eq. (8) to (13) clearly correspond to an elastic problem where the dependent variables and elastic constants are functions of the Laplace Transform parameter p.

The elastic equivalents of (8) and (9) are:

$$s_{ij} = 2G\, e_{ij}$$

$$\sigma_{kk} = 3K\, \epsilon_{kk}$$

If a solution to the associate elastic problem can be found then the viscoelastic solution is derived by applying the inverse Laplace Transform to the elastic solution after substitution of $G_1 p$ for 2G and $G_2 p$ for 3K.

The method of solution of the isothermal problem is therefore clearly defined.

Influence of Temperature on the Mechanical Properties

It has been established by experiments that increase in temperature accentuates creep and relaxation rates. that is, strains increase and stresses decay faster with time.

The direct experimental determination of the dependence of these rates on the current temperature is an exceedingly difficult task, as one can appreciate.

However, a wide class of morphous polymers obeying the linear viscoelastic law exhibit a simple property with change of temperature. This property has been utilized to derive the dependence of the mechanical properties on temperature for such a class. This property is illustrated in Fig. 1.

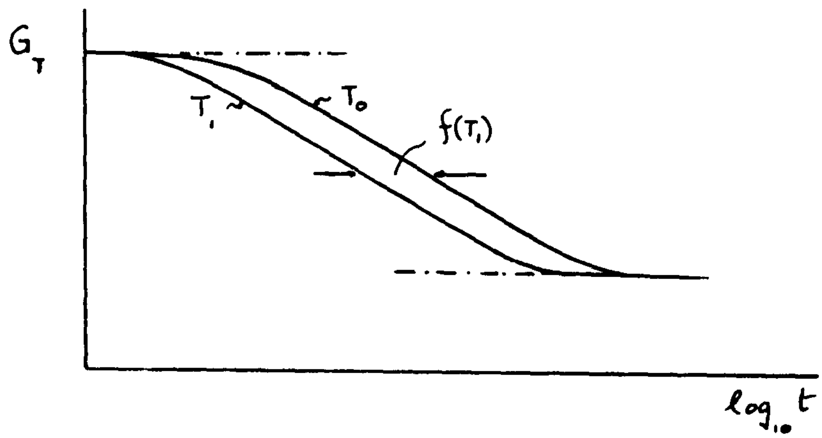


Figure 1

In this figure the relaxation modulus is plotted against logt for different constant uniform temperatures.

It is seen that the effect of a temperature increase, is a shift of the whole relaxation curve to the left, this shift being a function of temperature.

Let

$$\log t = u \tag{14}$$

$$t = e^{u} \tag{15}$$

then $\overset{*}{G}(u) = G\{\log t\}$ is obtained from G(t) by the transformation

$$G^{*}(u) = G(e^{u}) \tag{16}$$

the star is used to indicate the change in functional dependence.

In view of Fig. 1

$$G^{*}_{T_1}(u) = G^{*}_{T_0}\{u + f(T_1)\} = G_{T_0}\{e^{u + f(T_1)}\} \tag{17}$$

$$G^{*}_{T_1}(u) = G_{T_0}\{t\, e^{f(T_1)}\} \tag{18}$$

$$f(T_0) = 0 \qquad f(T) > 0 \quad \text{for} \quad T > T_0$$

We now put

$$e^{f(T_1)} = a(T_1) \tag{19}$$

Then

$$G_{T_1}(t) = G_{T_0}\{a(T_1)\, t\} \tag{20}$$

and generally

$$G_{T}(t) = G_{T_0}\{a(T)\, t\} \tag{21}$$

We now define a "reduced time" ξ such that

$$\xi = a(T)\,t \tag{22}$$

Then finally

$$G_T(t) = G_{T_o}(\xi) \tag{23}$$

Thus knowing G at some uniform reference temperature T_o, G at any other uniform temperature can be found.

From (23)

$$\frac{dG_T}{dt} = \frac{d\,G_{T_o}(\xi)}{d\xi}\left(\frac{d\xi}{dt}\right) = \left.\frac{dG_{T_o}}{dt}\right|_{t=\xi} a(T) \tag{24}$$

Eq. (24) gives a relation between the relaxation rates at temperature T as compared to To.

For instance at t = 0, ξ = 0

$$\left.\frac{dG_T}{dt}\right|_{t=0} = \left.\frac{dG_{T_o}}{dt}\right|_{t=0} a(T) \tag{25}$$

Note that $\xi \geq t$ for all temperature histories since $a(T) = e^{f(T)} \geq 0$ because of (17).

From (22) and (23)

$$G_T(o) = G_{T_o}(o) \tag{26}$$

Eq. (26) implies that the elastic (initial) response of the material is unaffected by temperature changes. Also in view of (26) and (23)

$$G_T(t-t_o) = G_{T_o}(\xi-\xi_o) \tag{27}$$

The constitutive relations at temperature T take the form:

$$S_{ij} = \int_o^t G_{1T}(t-\tau)\frac{\partial e_{ij}}{\partial \tau}d\tau = \int_o^t G_1(\xi-\xi')\frac{\partial e_{ij}}{\partial \tau}d\tau \tag{28}$$

where $\xi' = a_T \tau$, and G_1 is referred to the reference temperature To. Similarly,

$$\sigma_{kk} = \int_o^t G_2(\xi-\xi')\frac{\partial \epsilon_{kk}}{\partial \tau}d\tau \tag{29}$$

assuming a stress free field in the presence of the uniform temperature T. Experimental observations as well as thermodynamic considerations indicate that the same shift factor applies both to G_1 and G_2.

Steady Non-uniform Temperature

Relations (28) and (29) can be immediately generalized to steady non-uniform temperatures, where now both a_T and, hence ξ are functions of the space variables .

In an explicit form (28) and (29) become:

$$s_{ij} = \int_0^t G_1\{a_T(x_k)(t-\tau)\}\frac{\partial e_{ij}}{\partial \tau}\,d\tau \tag{30}$$

and

$$\sigma_{kk} = \int_0^t G_2\{a_T(x_k)(t-\tau)\}\frac{\partial}{\partial \tau}\{\epsilon_{kk} - 3\alpha_0\,\Theta(x_k)\}\,d\tau \tag{31}$$

where

$$\Theta = \frac{1}{\alpha_0}\int_{T_0}^{T} \alpha(T)\,dT \tag{32}$$

It is noteworthy that L.T.* is still applicable to (30) and (31) i.e.

$$\bar{s}_{ij} = \frac{p}{a_T(x_k)}\,\bar{G}_1\left\{\frac{p}{a_T(x_k)}\right\}\bar{e}_{ij} \tag{33}$$

---

* Henceforth L.T. will signify Laplace Transform.

$$\bar{\sigma}_{kk} = \frac{p}{a_T(x_k)} \bar{G}_2\left\{\frac{p}{a_T(x_k)}\right\}\left\{\bar{\epsilon}_{kk} - 3\alpha_0 \bar{\Theta}\right\} \tag{34}$$

However, in the resulting associate elastic problem the associate "elastic constants" are now functions of the space variables. Closed form solution to such a problem is rather unlikely, except for the simplest geometries, and resort to numerical procedures. is inevitable.

Temperature as Fun:tion of Space Variables and Time

The constitutive relations in the presence of transient temperature fields, have been formulated* as a generalization of Eq. (28) and (29), and take the following form :

$$s_{ij} = \int_0^t G_1(\xi - \xi') \frac{\partial e_{ij}}{\partial \tau} d\tau \tag{35}$$

$$\sigma_{kk} = \int_0^t G_2(\xi - \xi') \frac{\partial}{\partial \tau} \left\{\epsilon_{kk} - 3\alpha_0 \Theta\right\} d\tau \tag{36}$$

where now

$$\xi = \int_0^t a_T\left\{T(x_k, t)\right\} dt = \xi(x_k, t) \tag{37}$$

$$\xi' = \xi(x_k, \tau) \tag{38}$$

---

* See Ref [3]

Relations (35) and (36) though still relatively simple are no longer in a convolution form and L.T. is inapplicable, with respect to t.

A transformation has been suggested that, apparently, overcomes this difficulty.

Let

$$\xi = \xi(x_K, t) \tag{39}$$

$$\zeta_i = \zeta_i(x_K) \equiv x_i \tag{40}$$

It must be noted that since a_T is non-negative $\xi(x_K,t)$ is a monotomically increasing function of time for any x_K .

Hence relation (37) may be inverted in the form:

$$t = g(x_K, \xi) \tag{41}$$

Then

$$s_{ij}(x_K, t) = s_{ij}\{x_K, g(x_K, \xi)\} = \hat{s}_{ij}(x_K, \xi) \tag{42}$$

and similarly for other functions.

In view of (39), (40), (41) and (42), relations (35) and (36) take the form:

$$\hat{s}_{ij} = \int_0^{\xi} G(\xi - \xi') \frac{\partial \hat{e}_{ij}}{\partial \xi'} d\xi' \tag{43}$$

$$\hat{\sigma}_{kk} = \int_0^t G_2(\xi - \xi') \frac{\partial}{\partial \xi'} \left\{ \hat{\epsilon}_{kk} - 3\kappa_0 \hat{\Theta} \right\} d\xi' \tag{44}$$

Eq. (43) and (44) are now convolution integrals in the ξ variable, and hence L.T. is again applicable in this particular variable.

This transformation, however, modifies the field equations as well. For the sake of argument let a function

$$f = f(x_k, t) \tag{45}$$

Then

$$f_{,i} = \left. \frac{\partial f}{\partial x_i} \right|_{t,\, x_k (k \neq i)} \tag{46}$$

In terms of the new variables

$$f_{,k} = \frac{\partial f}{\partial \zeta_i} \frac{\partial \zeta_i}{\partial x_k} + \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x_k} \qquad \text{(i. summed)} \tag{47}$$

or,

$$\left. \frac{\partial f}{\partial x_k} \right|_{t,\, x_i (i \neq k)} = \left. \frac{\partial \hat{f}}{\partial x_k} \right|_{\xi,\, x_i (i \neq k)} + \left. \frac{\partial \hat{f}}{\partial \xi} \right|_{x_k} \left. \frac{\partial \xi}{\partial x_k} \right|_{t} \tag{48}$$

The equilibrium relation

$$\sigma_{ij,j} + f_i = 0 \tag{49}$$

becomes:

$$\hat{\sigma}_{ij,j}\Big|_{\xi} + \frac{\partial \hat{\sigma}_{ij}}{\partial \xi}\frac{\partial \xi}{\partial x_j}\Big|_t + \hat{f}_i = 0 \tag{50}$$

Note that

$$\frac{\partial \xi}{\partial x_i}\Big|_t = F(x_i, t) = F\{x_j, g(x_j, \xi)\} = \hat{F}(x_k, \xi) \tag{51}$$

Since L.T. of an ordinary product of two functions is not always defined L.T. is inapplicable to Eq. (50).

Obviously the above transformation simplifies the constitive relations but complicates the field equations.

This transformation, however, is useful in the case where T = T(t) only. Then,

$$\xi = \xi(t) \tag{52}$$

and (50) reduces to

$$\hat{\sigma}_{ij,j}\Big|_{\xi} + \hat{f}_i = 0 \tag{53}$$

Now L.T. of the field equations and constitutive relation can be taken with respect to ξ and the viscoelastic problem reduces to an associate elastic problem as in the isothermal case.

We summarize as follows:

(a) Isothermal case: L.T. is applicable with respect to t and the viscoelastic problem reduces to an associate elastic problem.

(b) Temperature space dependent but steady: L.T. applicable with respect to t. Viscoelastic problem reduces to an associate elastic problem with material constants as functions of the space variables.

(c) Temperature uniform but time dependent: L.T. applicable with respect to the reduced variable ξ . Viscoelastic problem reduces to an associate elastic problem as in case (a).

(d) Temperature both space and time dependent: L.T. inapplicable. Associate elastic problem does not exist.

A certain amount of simplification is achieved in cases where the dilatational response of the material is elastic. If the material is also thermorheologically simple, then the dilatational response remains unaffected by temperature changes (since when the function is shifted it merely reproduces itself).

The dilatational behavior of the material is now given in the form

$$\sigma_{kk} = 3K \left\{ \epsilon_{kk} - 3\alpha_0 \Theta \right\} \tag{54}$$

where K is now independent of time and temperature.

An Alternative Integral Form of the Constitutive Equations

It must be emphasized that Eq. (1) and (54) are not a unique integral representation of the viscoelastic constitutive laws. One, for instance, may make use of the "tension modulus" E and "poissons ratio" ν , where from a purely formal standpoint E(t) is the stress response of a tensile specimen to a constant axial unit strain, and ν (t) is the corresponding lateral strain response to the same strain.

Under isothermal conditions the stress-strain relations for direct stresses and strains become,

$$\int^{t} E(t-\tau)\,\frac{\partial \epsilon_{11}}{\partial \tau}\,d\tau = \sigma_{11} - \int_{0}^{t} \nu(t-\tau)\,\frac{\partial}{\partial \tau}(\sigma_{22}+\sigma_{33})\,d\tau \qquad (55)$$

where suffices 1, 2, 3 are cyclically interchangeable.

For a material with elastic dilatational response the formal equivalence of (54) and (55) can be found by taking L.T.

Then (1) and (54) become:

$$\bar{s}_{ij} = p\bar{G}\,\bar{e}_{ij} \qquad (56)$$

$$\bar{\sigma}_{kk} = 3K\,\bar{\epsilon}_{kk} \qquad (57)$$

whereas (55) now is:

$$p\bar{E}\,\bar{\epsilon}_{11} = \bar{\sigma}_{11} - \bar{\nu}\,p(\bar{\sigma}_{22}+\bar{\sigma}_{33}) \qquad (58)$$

or

$$p \bar{E} \bar{\epsilon}_{11} = (1 + \bar{\nu} p) \bar{\sigma}_{11} - p \bar{\nu} \bar{\sigma}_{kk} \tag{59}$$

$$\bar{\epsilon}_{11} = \frac{1 + p\bar{\nu}}{p\bar{E}} \bar{\sigma}_{11} - \frac{p\bar{\nu}}{p\bar{E}} \bar{\sigma}_{kk} \tag{60}$$

Comparison of (56), (57) and (60) and a few algebraic manipulations yield

$$\bar{E} = \frac{9K\bar{G}}{6K + p\bar{G}} \tag{61}$$

$$\bar{\nu} = \frac{1}{p} \frac{3K - p\bar{G}}{6K + p\bar{G}} \tag{62}$$

$$\bar{G} = \frac{\bar{E}}{1 + p\bar{\nu}} \tag{63}$$

$$K = \frac{p\bar{E}}{3(1 - 2p\bar{\nu})} \tag{64}$$

Hence in terms of E and ν the complete stress-strain relations are:

$$\int_0^t E(t-\tau)\frac{\partial \epsilon_{ij}}{\partial \tau}\,d\tau \; =$$

$$\sigma_{ij} + \int_0^t \nu(t-\tau)\frac{\partial}{\partial \tau}(\sigma_{ij}-\sigma_{kk})\,d\tau \quad (i=j) \qquad (65)$$

$$\sigma_{ij} = \int_0^t G(t-\tau)\frac{\partial \epsilon_{ij}}{\partial \tau}\,d\tau \qquad (i \neq j) \qquad (66)$$

Since all properties should obey the same shift law (65) and (66) can be generalized for the non-isothermal case i.e.

$$\int_0^t E(\xi-\xi')\frac{\epsilon_{ij}}{\partial \tau}\,d\tau \; =$$

$$\sigma_{ij} + \int_0^t \nu(\xi-\xi')\frac{\partial}{\partial \tau}(\sigma_{ij}-\sigma_{kk})\,d\tau \quad (i=j) \qquad (67)$$

and naturally

$$\sigma_{ij} = \int_0^t G(\xi-\xi')\frac{\partial \epsilon_{ij}}{\partial \tau}\,d\tau \qquad (i \neq j) \qquad (68)$$

In the subsequent analysis we shall have occasion to use both constitutive representations.

CHAPTER I

Thermal Stresses In Viscoelastic Solids With Material Properties Exhibiting Thermorheologically Simple Temperature Dependence.

1.1 Introduction

It was pointed out in the general introduction that when the temperature field is transient and space dependent, elimination of the time dependence from the constitutive and field equations simultaneously has been impossible by means of L.T. or any other exact transformation.

Part of the present chapter will be devoted to developing approximate techniques by which time dependence may be eliminated, thus facilitating the solution of the relevant equations. In fact two such techniques are given which are not dependent on the particular geometry of the body at hand.

However, other approximations, such as material incompressibility or effective constant Poisson's ratio, that are suited to a particular geometry, such as the hollow cylinder, are also used with advantage. The inapplicability of the principle of superposition - except in a very restricted sense - robs the analysis of a powerful tool, and limits the chances of development of an exact general theory.

1.2 General Approximate Techniques

An approximation rendering L.T. applicable

Consider the deviatoric constitutive relation (35) in the presence of a transient non-uniform temperature field.

$$S_{ij} = \int_0^t G(\xi - \xi') \frac{\partial e_{ij}}{\partial \tau} d\tau \qquad (1.2.1)$$

where the suffix is omitted.

Figure (1.2.1) shows the variation of ξ with time for monotorically increasing and decreasing temperature.

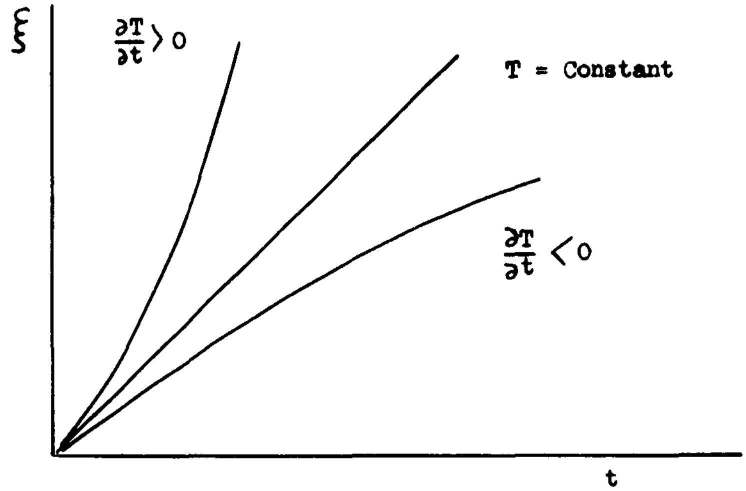


Figure (1.2.1)

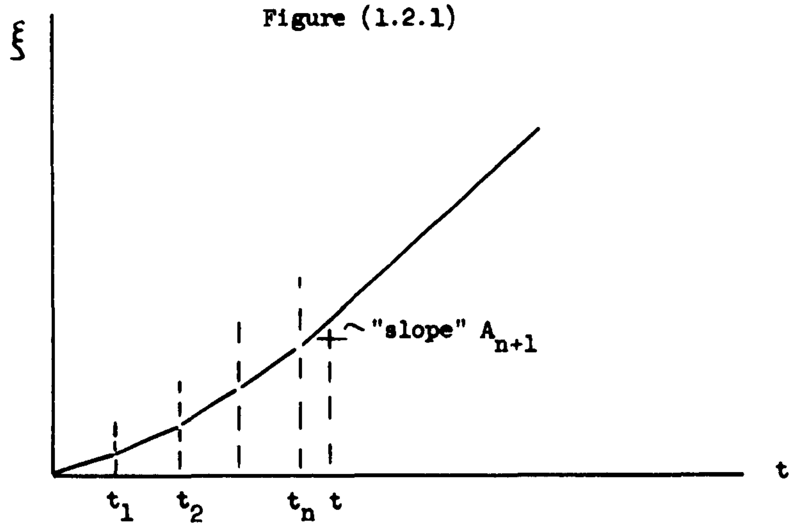


Figure (1.2.2)

The present technique hinges on the piecewise linearization of over small time intervals,

Then in any typical interval K,

$$\xi = A_K t + B_K \quad , t_{K-1} \leq t \leq t_K \qquad (1.2.2)$$

where A_K is the "slope" of ξ in the K^{th} interval (Fig. 1.2.2)

We now assume that by the present procedure, we have found the expressions for $e_{ij}(t)$ for each interval up to t_n , and call the expression for e_{ij} in the interval K by $e_{ij}^{K}(t)$ where $t_{K-1} \leq t \leq t_K$.

By introducing (1.2.2) and carrying out the integration in (1.2.1) over all intervals for $t_n \leq t \leq \infty$ and assuming that $\xi(t)$ continues as a straight line for $t_n \leq t \leq \infty$ we obtain:

$$S_{ij}(t) = \sum_{K=1}^{n} \int_{t_{K-1}}^{t_K} G\left\{A_{n+1}t - A_K\tau + B_{n+1} - B_K\right\} \frac{\partial e_{ij}^{K}}{\partial \tau} d\tau$$

$$+ \int_{t_n}^{t} G\left\{A_{n+1}(t-\tau)\right\} \frac{\partial e_{ij}^{n+1}}{\partial \tau} d\tau \qquad (1.2.3)$$

Let

$$I_{n+1}(t) = \int_{t_n}^{t} G\left\{A_{n+1}(t-\tau)\right\} \frac{\partial e_{ij}^{n+1}}{\partial \tau} d\tau \qquad (1.2.4)$$

$$\underline{I}_{\kappa}(t) = \int_{t_{\kappa-1}}^{t_\kappa} G\left\{A_{n+1}t - A_\kappa \tau + B_{n+1} - B_\kappa\right\} \frac{\partial e^{\kappa}_{ij}}{\partial \tau} d\tau \qquad (1.2.5)$$

Then

$$s_{ij}(t) = \sum_{\kappa=1}^{n} \underline{I}_{\kappa}(t) + \underline{I}_{n+1}(t). \qquad (1.2.6)$$

We now make the following transformation

$$x = t - t_n \qquad (1.2.7)$$

where now $\quad 0 \leq x \leq \infty$

and in the last integral $\underline{I}_{n+1}$

$$y = \tau - t_n \qquad (1.2.8)$$

where $\quad 0 \leq y \leq x$

In view of (1.2.7) and (1.2.8)

$$\overset{n+1}{e_{ij}}(\tau) = \overset{n+1}{e_{ij}}(y + t_n) = \underset{\sim}{e}_{ij}(y) \qquad (1.2.9)$$

$$\overset{n+1}{e_{ij}}(t) = \overset{n+1}{e_{ij}}(x + t_n) = \underset{\sim}{e}_{ij}(x) \qquad (1.2.9a)$$

On substitution in (1.2.4) we obtain:

$$\underline{I}_{n+1}(x) = \int_0^x G\left\{A_{n+1}(x-y)\right\} \frac{\partial \overset{n+1}{e_{ij}}(y)}{\partial y} dy \qquad (1.2.10)$$

Now taking L.T. of (1.2.10) and letting

$$\bar{\underline{I}}_{n+1}(p) = L.T.\left\{\underline{I}_{n+1}(x)\right\} \qquad (1.2.11)$$

we obtain

$$\bar{\underline{I}}_{n+1}(p) = \frac{1}{A_{n+1}} \bar{G}\left(\frac{p}{A_{n+1}}\right)\left\{p \overset{n+1}{\bar{e}_{ij}} - \overset{n+1}{e_{ij}}(x=0)\right\} \qquad (1.2.12)$$

But

$$\overset{n+1}{e_{ij}}(x=0) = \overset{n+1}{e_{ij}}(t_n) = \overset{n}{e_{ij}}(t_n) \qquad (1.2.13)$$

in view of the continuity of strain at $t = t_n$ and $\overset{n}{e_{ij}}(t_n)$ is known from the previous solution.

Thus (1.2.12) becomes

$$\bar{\underline{I}}_{n+1}(p) = \frac{1}{A_{n+1}} \bar{G}\left(\frac{p}{A_{n+1}}\right)\left\{p \overset{n+1}{\bar{e}_{ij}} - \overset{n}{e_{ij}}(t_n)\right\} \qquad (1.2.14)$$

The expression (1.2.4) for I_K with the change of variable (1.2.7) becomes:

$$I_\kappa(x) = \int_{t_{\kappa-1}}^{t_\kappa} G\left\{A_{n+1}(x+t_n) - A_\kappa \tau + B_{n+1} - B_\kappa\right\} \frac{\partial e_{ij}^{\kappa}}{\partial \tau}\, d\tau$$

$$= \int_{t_{\kappa-1}}^{t_\kappa} G\left\{A_{n+1}x + A_{n+1}t_n - A_\kappa \tau + B_{n+1} - B_\kappa\right\} \frac{\partial e_{ij}^{\kappa}}{\partial \tau}\, d\tau \quad (1.2.15)$$

since

$$A_{n+1}\, t_n + B_{n+1} = \xi_n \quad (1.2.16)$$

$$I_\kappa(x) = \int_{t_{\kappa-1}}^{t_\kappa} G\left\{A_{n+1}x + \xi_n - B_\kappa - A_\kappa \tau\right\} \frac{\partial e_{ij}^{\kappa}}{\partial \tau}\, d\tau \quad (1.2.17)$$

Taking L.T. of (1.2.16) with respect to x:

$$\bar{I}_\kappa(p) = \int_0^\infty e^{-px} \left\{ \int_{t_{\kappa-1}}^{t_\kappa} G\left\{A_{n+1}x + \xi_n - B_\kappa - A_\kappa \tau\right\} \frac{\partial e_{ij}^{\kappa}}{\partial \tau}\, d\tau \right\} dx$$

$$= \int_{t_{\kappa-1}}^{t_\kappa} \frac{\partial e_{ij}^{\kappa}}{\partial \tau} \left\{ \int_0^\infty e^{-px} G\left\{A_{n+1}x + \xi_n - B_\kappa - A_\kappa \tau\right\} dx \right\} d\tau$$

Or

$$\bar{I}_\kappa(p) = \frac{1}{A_{n+1}}\, e^{\frac{\xi_n - B_\kappa}{A_{n+1}} p}\, \bar{G}\left(\frac{p}{A_{n+1}}\right) \int_{t_{\kappa-1}}^{t_\kappa} e^{-\frac{A_\kappa}{A_{n+1}} p\tau} \frac{\partial e_{ij}^{\kappa}}{\partial \tau}\, d\tau \quad (1.2.18)$$

Since $e_{ij}^{\kappa}(t)$ for $t_{\kappa-1} \leq t \leq t_\kappa$ is known from a previous step of the solution, $\bar{I}_\kappa$ is known function of p found from Eq. (1.2.17).

Thus calling

$$s_{ij}(t) = s_{ij}(\lambda + t_n) = \underset{\sim}{s}_{ij}(\lambda) \tag{1.2.19}$$

and taking L.T. of (1.2.3) and in view of (2.3.14) and (1.2.18) and (1.2.19) we obtain:

$$\bar{\underset{\sim}{s}}_{ij}(p) = \frac{1}{A_{n+1}} \bar{G}\left(\frac{p}{A_{n+1}}\right) \left\{ e^{\frac{\xi_n}{A_{n+1}} p} \sum_{K=1}^{n} e^{-\frac{\theta_K}{A_{n+1}} p} \int_{t_{K-1}}^{t_K} e^{-\frac{A_K}{A_{n+1}} p\tau} \frac{\partial \overset{K}{e}_{ij}}{\partial \tau} d\tau - \overset{n}{e}_{ij}(t_n) \right\}$$

$$+ \frac{p}{A_{n+1}} \bar{G}\left(\frac{p}{A_{n+1}}\right) \bar{\underset{\sim}{e}}{}^{n+1}_{ij}(p) \tag{1.2.20}$$

Eq. (1.2.20) is a linear relationship between the transformed functions $\bar{\underset{\sim}{s}}_{ij}(p)$ and $\bar{\underset{\sim}{e}}{}^{n+1}_{ij}(p)$. If $s_{ij}(t)$ is known then $\bar{\underset{\sim}{s}}_{ij}(p)$ is also known and thus $\underset{\sim}{e}{}^{n+1}_{ij}(\lambda)$ and hence $\underset{\sim}{e}{}^{n+1}_{ij}(t)$ can be found from (1.2.20) by an inverse L.T. Therefore the strain $e^{n+1}_{ij}(t)$ is known as a function of time in the interval $t_n \leq t \leq \infty$.

This solution, however, is utilized only for $t_n \leq t \leq t_{n+1}$. For $t_{n+1} < t$ the curve $\xi(t)$ is continued with a new slope as a straight line to infinite, and the above procedure is repeated.

Eq. (1.2.20) can be used in conjunction with the solution of the boundary value problem in the following manner.

In the interval $t_n \leq t \leq \infty$ (1.2.20) is written in the form:

$$\bar{\underset{\sim}{s}}_{ij}(p) = \bar{F}_1(p) + \frac{p}{A_{n+1}} \bar{G}_1\left(\frac{p}{A_{n+1}}\right) \bar{\underset{\sim}{e}}{}^{n+1}_{ij}(p) \tag{1.2.21}$$

where $\bar{F}_1(p)$ is a known function. A similar expression may be derived for the hydrostatic constitutive relation, i.e.

$$\bar{\underset{\sim}{\sigma}}_{kk}(p) = \bar{F}_2(p) + \frac{p}{A_{n+1}} \bar{G}_2\left(\frac{p}{A_{n+1}}\right)\left\{ \bar{\underset{\sim}{\epsilon}}_{kk}^{\,n+1} - 3\alpha_0 \bar{\underset{\sim}{\Theta}}^{\,n+1} \right\} \quad (1.2.22)$$

The field equations for the current interval may also be put in the form

$$\left.\begin{aligned} \bar{\underset{\sim}{\sigma}}_{ij,j} + \bar{\underset{\sim}{f}}_i &= 0 \\ \bar{\underset{\sim}{\epsilon}}_{ij} &= \frac{1}{2}\left\{ \bar{\underset{\sim}{u}}_{i,j} + \bar{\underset{\sim}{u}}_{j,i} \right\} \\ \bar{\underset{\sim}{\sigma}}_{ij}\, n_j &= \bar{\underset{\sim}{T}}_i \quad \text{on } S_1 \\ \bar{\underset{\sim}{u}}_i &= \bar{\underset{\sim}{u}}_i^{(S_2)} \quad \text{on } S_2 \end{aligned}\right\} \quad (1.2.23)$$

Eq. (1.2.21), (1.2.22) and (1.2.23) are necessary and sufficient for the solution of the non-isothermal viscoelastic boundary value problem in the transformed plane.

This technique has been used with encouraging results in the case of the slab, see Ref. [10] . In the same reference, the problem has been formulated where the constitutive equations are given in the form of differential operators.

Reduction of the Constitutive Integral Equations to a Set of Simultaneous Algebraic Equations

Consider the following constitutive equation of the Volterra type

$$s = \int_0^t G_1\left\{\xi(x_K, t) - \xi(x_K, \tau)\right\} \frac{\partial e}{\partial \tau}\, d\tau \tag{1.2.24}$$

where s is the deviatoric stress tensor and e is the deviatoric strain tensor.

For simplicity denote

$$G_1\left\{\xi(x_K, t) - \xi(x_K, \tau)\right\} \equiv G_1(x_K, t, \tau) \tag{1.2.25}$$

then (1.2.24) becomes

$$s = \int_0^t G(x_K, t, \tau) \frac{\partial e}{\partial \tau}\, d\tau \tag{1.2.26}$$

For x_K fixed, $e(t)$ can be approximated by a piece-wise linear function of t (Figure 2.1.3).

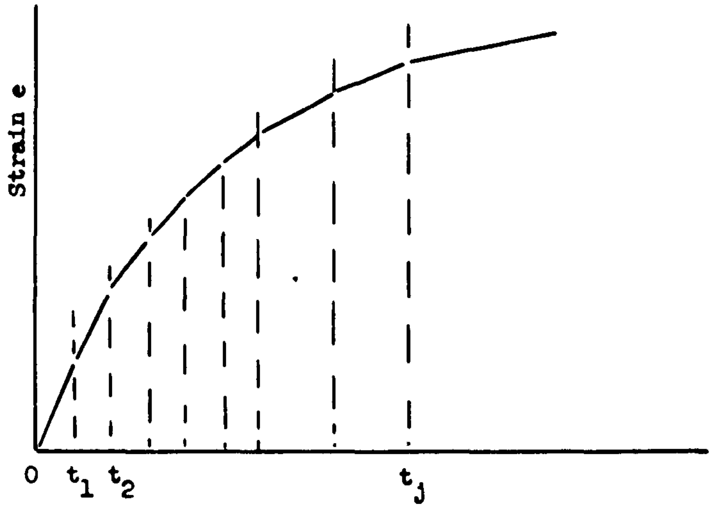


Figure (1.2.3)

LINEAR PIECEWISE VARIATION

Let A_n represent the "slope" in the n^{th} interval; and S_n , e_n the values of s and e at the end of the n^{th} interval.

Then,

$$S_n = A_1 \int_0^{t_1} G_1(x_k, t, \tau)\, d\tau + \cdots$$

$$\cdots + A_r \int_{t_{r-1}}^{t_r} G_1(x_k, t, \tau)\, d\tau + \cdots$$

$$\cdots + A_n \int_{t_{n+1}}^{t_n} G(x_k, t, \tau)\, d\tau \qquad (1.2.27)$$

Let

$$\int_{t_{r-1}}^{t_r} G_1(x_k, t, \tau)\, d\tau = G_{1,nr}(x_k)\, \Delta t_r \qquad (1.2.28)$$

where

$$\Delta t_r = t_r - t_{r-1} \qquad (1.2.29)$$

Then

$$s_n = \sum_{r=1}^{n} G_{1,nr}\, a_r \qquad (1.2.30)$$

where

$$a_r = A_r \Delta t_r \qquad (1.2.31)$$

or in matrix form

$$\{s\} = [G_1] \{a\} \qquad (1.2.32)$$

where $\left[G_1\right] = \begin{bmatrix} G_{1,11} & & & \\ G_{1,21} & G_{1,22} & & \\ G_{1,31} & G_{1,32} & G_{1,33} & \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ G_{1,n1} & \cdots & & G_{1,nn} \end{bmatrix}$ (1.2.33)

Note that $\left[G_1\right]$ is a triangular matrix.

Since

$$a_r = e_r - e_{r-1} \tag{1.2.34}$$

we have the relation

$$\{e\} = [H]\{a\} \tag{1.2.35}$$

where $[H] = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$ (1.2.36)

From (1.2.32)

$$\{a\} = [H]^{-1}\{s\} \tag{1.2.37}$$

and

$$e = [H]\left[G_1\right]^{-1}\{s\} \tag{1.2.38}$$

also

$$\{s\} = [G_1][H]^{-1}\{e\} \qquad (1.2.39)$$

$[G_1]$ being triangular, $[G_1]^{-1}$ is very easy to find. Eq. (1.2.38) is a numerical solution of the Volterra equation (2.3.24). However, more will be said about this, in the closing parts of this chapter.

Application to the Non-isothermal Viscoelastic Boundary Value Problem

Equation (1.2.32) can serve as a basis for solving the non-isothermal boundary value problem for all t by solving consequentively for small finite time intervals.

In this way the time dependence is eliminated and the problem reduces to the solution of a related elastic problem with "initial strains" and spatially dependent elastic properties.

In the general problem, parallel to (1.2.32) there will be a dilatational stress-strain relation which again can be put in the form

$$\{\sigma_{kk}\} = [G_2][H]^{-1}\{\epsilon_{kk} - 3\alpha_0\Theta\} \qquad (1.2.40)$$

also

$$\{\epsilon_{kk}\} = [H][G_2]^{-1}\{\sigma_{kk}\} + 3\alpha_0\{\Theta\} \qquad (1.2.41)$$

we call

$$[G_2][H]^{-1} = [G_2^*] \qquad (1.2.42)$$

and

$$[G_1][H]^{-1} = [G_1]^* \tag{1.2.43}$$

Where $[G_1^*]$ and $[G_2]^*$ are triangular matrices.

In the first interval the stress-strain relations become:

$$S_1 = G^*_{1,11}\, e_1 \tag{1.2.44}$$

$$\sigma_1 = G^*_{2,11}\left[\epsilon_1 - 3\alpha_0 \Theta_1\right] \tag{1.2.45}$$

$$(\sigma_{KK} \equiv \sigma \;, \quad \epsilon_{KK} \equiv \epsilon)$$

Clearly (1.2.44) and (1.2.45) together with the equilibrium equations, strain displacement relation, and boundary conditions constitute an elastic problem with spatially dependent elastic constants.

In the second interval we get

$$S_2 = G^*_{1,21}\, e_1 + G^*_{1,22}\, e_2 \tag{1.2.46}$$

$$\sigma_2 = G^*_{2,21}\left[\epsilon_1 - 3\alpha_0 \Theta_1\right] + G^*_{2,22}\left[\epsilon_2 - 3\alpha_0 \Theta_2\right] \tag{1.2.47}$$

We now have an elastic problem with "initial" known strains e_1, ϵ_1 and $3\alpha_0 \Theta_1$. Therefore in any typical interval we can solve for the stresses in that interval in terms of the temperature in that same interval, and the stresses, strain & temperature in all previous intervals. The solution for all time can be determined in this

systematic way.

It is interesting to note that moving boundaries can be dealt with by means of the above method, in the light of small deformation theory. This is simply done by determining the new boundary from the solution in the previous interval and assuming the boundary to remain fixed in the current interval. This naturally is only approximately true.

The above approach provides a systematic method through which the general viscoelastic boundary value problem can be solved by numerical means.

A Variation on the Previous Technique

An alternative approach is to consider $e(t)$ as an escalator function of time. See Fig. (1.2.4)

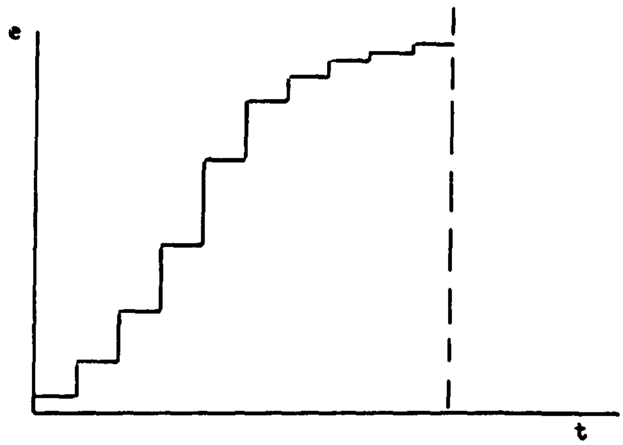


Figure (1.2.4)

Then

$$e(t) = \sum_{\tau} a_{\tau} \, H(t - t_{\tau}) \qquad (\tau = 0, 1, 2 \ldots) \qquad (1.2.48)$$

and

$$\frac{\partial e}{\partial t} = \sum_{r} a_{r}\, \delta(t - t_{r}) \tag{1.2.49}$$

Substituting (1.2.49) in (1.2.26) we obtain

$$s(t) = \sum_{r} a_{r} \int_{0}^{t} G_{1}(x_{k}, t, \tau)\, \delta(\tau - t_{r})\, d\tau = \sum_{r} a_{r} G_{1}(x_{k}, t, t_{r}) \tag{1.2.50}$$

Thus

$$s_{n} = \sum_{r=0}^{n} a_{r} G_{1}(x_{k}, t_{n}, t_{r}) = \sum_{r=0}^{n} a_{r} G_{1,nr} \tag{1.2.51}$$

where

$$G_{1,nr} = G_{1}(x_{k}, t_{n}, t_{r}) \tag{1.2.52}$$

Eq. (1.2.51) is identical in form with Eq. (1.2.30), however, the coefficients $G_{1,nr}$ are now defined differently. The advantage of this particular formulation, at the expense of smaller time intervals, is that

$$G_{1,nn} = G_{1}(0) \tag{1.2.52}$$

for all n and all temperature variations, this simply being the elastic response of the material which the shift hypothesis renders independent of temperature.

Naturally, we have again in matrix form:

$$\{S\} = [G_1][H]^{-1}\{e\} \tag{1.2.53}$$

where now the elements of the column matrices $\{S\}$ and $\{e\}$ are the average values of the deviatoric stress and strains in the corresponding time intervals. Also,

$$\{\sigma\} = [G_2][H]^{-1}\{\epsilon - 3\alpha_0 \Theta\} \tag{1.2.54}$$

Again, in the first time interval we have Eq's (1.2.44) and (1.2.45). $G^*_{1,11}$ and $G^*_{2,11}$, however, are now constants and the viscoelastic problem is reduced to an elastic problem.

In view of (1.2.52) the viscoelastic problem is repeatedly reduced to an elastic problem in <u>all</u> the intervals except that for intervals, subsequent to the first, known "initial strains" will also be present. The power of this technique does not need emphasizing.

1.3 Solution By Assuming Mechanical Incompressibility

This assumption has been utilized in the past 2 to obtain solutions to viscoelastic boundary value problems. It is based on the hypothesis that, where the dilatational response is elastic, the bulk modulus K is of exceedingly high order of magnitude so that in the expression

$$\epsilon = \frac{\sigma}{3K} + 3\alpha_0 \Theta \tag{1.3.1}$$

the $\frac{\sigma}{3K}$ term is negligible and hence

$$\epsilon = 3\alpha_0 \Theta \tag{1.3.2}$$

This simplification can lead in certain cases to a closed form solution - see sections 4 and 5a - however, unless K is in fact very large such a solution can only be approximate.

On the other hand, such a solution can serve a very good zero'th approximation to an iteration process.

For the sake of illustration we give an account of this process for a plane strain problem $\epsilon_z = 0$, however, the method is quite general. A solution is found for $\frac{\sigma}{3K} = 0$. From this solution, σ is calculated from

$$\sigma = \sigma_x + \sigma_y + \int_0^t \nu(\xi - \xi')\frac{\partial}{\partial\tau}(\sigma_x + \sigma_y)\,d\tau - \alpha_0 \int_0^t E(\xi - \xi')\frac{\partial\Theta}{\partial\tau}\,d\tau \tag{1.3.3}$$

This value of σ is substituted in (1.3.1) and K is taken as the actual modulus of the material. Then ϵ is found in terms of a new temperature Θ_1 where:

$$\Theta_1 = \Theta + \frac{\sigma_0}{9\alpha_0 K} \tag{1.3.4}$$

where σ_0 is the value found from the first solution and

$$\epsilon_1 = 3\alpha_0 \Theta_1 \tag{1.3.5}$$

A second solution is now found in terms of Θ_1 and the process is repeated to required accuracy.

1.4 Application to a Viscoelastic Cylinder Enclosed in an Elastic Case

Formulation of the Problem

Here we consider a hollow cylinder under plane strain subject to a radial transient temperature field. The cylinder is enclosed by a thin elastic shell which is assumed to be rigidly bonded to the cylinder (Fig. 1.4.1).

Under these conditions

$$T = T(r, t) \tag{1.4.1}$$

$$u_z = u_\theta = 0 \quad , \quad u_r = u(r, t) \tag{1.4.2}$$

$$\epsilon_{r\theta} = \epsilon_{rz} = \epsilon_{\theta z} = \epsilon_z = 0 \tag{1.4.3}$$

$$\epsilon_r = \frac{\partial u}{\partial r} \quad , \quad \epsilon_\theta = \frac{u}{r} \tag{1.4.4}$$

We now have the following compatibility relations in view of (1.4.2), (1.4.3) and (1.4.4).

$$\epsilon_r = \frac{\partial}{\partial r}(r\epsilon_\theta) \tag{1.4.5}$$

$$\epsilon = \frac{1}{r}\frac{\partial}{\partial r}(r^2\epsilon_\theta) \tag{1.4.6}$$

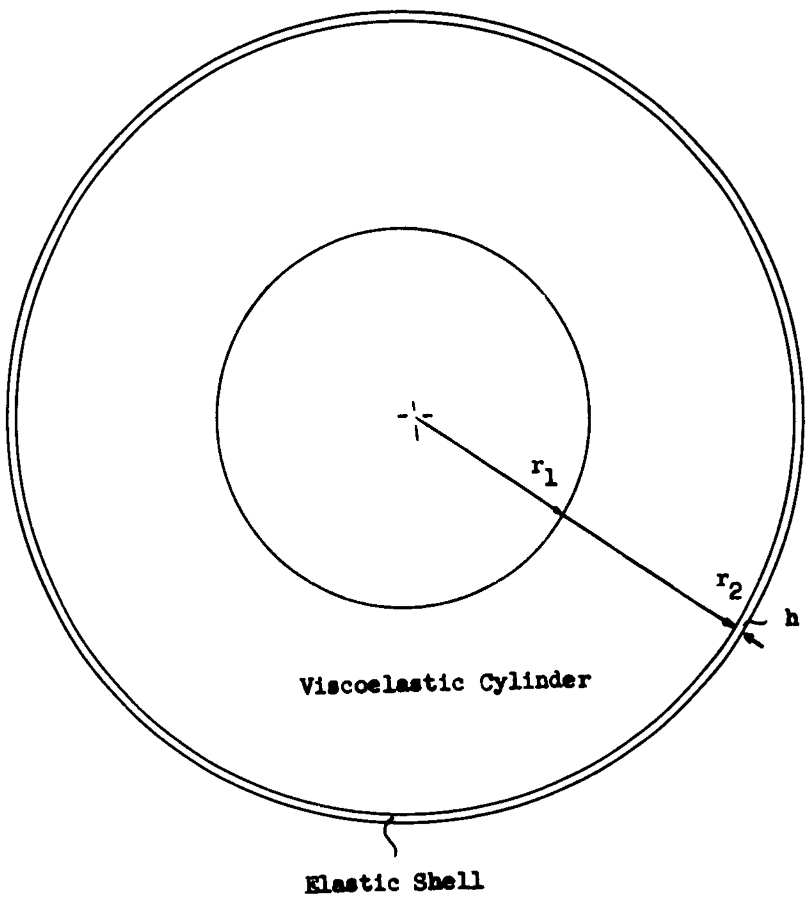


Figure (1.4.1)

$$\frac{\epsilon_r - \epsilon_\theta}{r} = \frac{e_r - e_\theta}{r} = \frac{\partial \epsilon_\theta}{\partial r}, \quad \epsilon = \epsilon_r + \epsilon_\theta \tag{1.4.7}$$

where e_r and e_θ are deviatoric strains.

The equilibrium relation to be satisfied is

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1.4.8}$$

Making use of Eq. (54) we get

$$\epsilon = \frac{\sigma}{3K} + 3\alpha_0 \Theta \tag{1.4.9}$$

Iteration Solution

We now empl9y the hypothesis that the material is elastically incompressible in dilatation by mechanical forces, [2], i.e.:

$$\frac{\sigma}{3K} = 0 \tag{1.4.10}$$

and use this assumption as the zero'th approximation to the solution of our problem, as in the case of the infinite slab.

Then from (1.4.6) and (1.4.9) and (1.4.10):

$$\epsilon = \frac{1}{r}\frac{\partial}{\partial r}(r^2 \epsilon_\theta) = 3\alpha_0 \Theta + \frac{\sigma}{3K} \tag{1.4.11}$$

Integrating

$$\epsilon_\theta = \frac{3\alpha_0}{r^2}\int_{r_2}^{r} \rho\, \Theta\, d\rho + \frac{1}{3r^2}\int_{r_2}^{r} \rho\left(\frac{\sigma}{K}\right) d\rho$$

$$+ \epsilon_\theta(r_2, t)\left(\frac{r_2}{r}\right)^2 \tag{1.4.12}$$

Making use of (1.4.10):

$$\epsilon_\theta = \frac{3\alpha_0}{r^2}\int_{r_2}^{r} \rho\, \Theta\, d\rho + \epsilon_\theta(r_2, t)\left(\frac{r_2}{r}\right)^2 \tag{1.4.13}$$

or

$$\epsilon_\theta = 3\alpha_0 \psi(r, t) + \epsilon_\theta(r_2, t)\left(\frac{r_2}{r}\right)^2 \tag{1.4.14}$$

where

$$\psi = \frac{1}{r^2}\int_{r_2}^{r} \rho\, \Theta\, d\rho \tag{1.4.15}$$

From Eq. (1.4.7) we obtain:

$$\frac{\partial \epsilon_\theta}{\partial r} = \frac{1}{r}(\epsilon - 2\epsilon_\theta) \tag{1.4.16}$$

which in view of (1.4.9) and (1.4.10) becomes;

$$\frac{\partial \epsilon_\theta}{\partial r} = \frac{1}{r}\left\{3\alpha_0 \Theta - 2\epsilon_\theta\right\} \qquad (1.4.17)$$

or

$$\frac{\partial \epsilon_\theta}{\partial r} = \frac{1}{r}\left\{3\alpha_0 \Theta - 6\alpha_0 \Psi(r,t) - 2\epsilon_\theta(r_2,t)\left(\frac{r_2}{r}\right)^2\right\} \qquad (1.4.18)$$

We now use (1.4.7), (1.4.8) and (30) to obtain

$$\frac{\partial \sigma_r}{\partial r} = -\int_0^t G(\xi-\xi')\frac{\partial}{\partial \tau}\left(\frac{\partial \epsilon_\theta}{\partial r}\right)d\tau \qquad (1.4.19)$$

which in view of (1.4.18) becomes

$$\frac{\partial \sigma_r}{\partial r} = -\frac{3\alpha_0}{r}\int_0^t G(\xi-\xi')\frac{\partial \Theta}{\partial \tau}d\tau + \frac{6\alpha_0}{r}\int_0^t G(\xi-\xi')\frac{\partial \Psi}{\partial \tau}d\tau$$

$$+ 2\frac{r_2^2}{r^3}\int_0^t G(\xi-\xi')\frac{\partial \epsilon_\theta(r_2,\tau)}{\partial \tau}d\tau \qquad (1.4.20)$$

Let

$$F(r,t) = \frac{3\alpha_0}{r}\int_0^t G(\xi-\xi')\frac{\partial}{\partial \tau}\left(2\Psi - \Theta\right)d\tau \qquad (1.4.21)$$

Then

$$\frac{\partial \sigma_r}{\partial r} = F(r,t) + \frac{2 r_2^2}{r^3} \int_0^t G(\xi-\xi') \frac{\partial \epsilon_\theta(r_2,\tau)}{\partial \tau} d\tau \qquad (1.4.22)$$

and

$$\sigma_r = \int_{r_1}^{r} F(\rho,t) d\rho + 2 r_2^2 \int_0^t \int_{r_1}^{r} \frac{1}{\rho^3} G(\xi-\xi') \frac{\partial \epsilon_\theta(r_2,\tau)}{\partial \tau} dr\, d\tau \qquad (1.4.23)$$

Obviously $\sigma_{r_1} = 0$ since $r = r_1$, is a free surface.

If at $r = r_2$ there is a rigid boundary $\epsilon_\theta(r_2,t) = 0$ and the complete solution is

$$\sigma_r = \int_{r_1}^{r} F(\rho,t) d\rho \qquad (1.4.24)$$

In the case of the boundary being an elastic shell σ_r and ϵ_θ for both cylinder and shell must be equal at the common boundary.

Let $\Theta_s(t)$ be the temperature of the shell assumed constant through the thickness. All parameters with suffix S refer to the elastic shell.

Then

$$\epsilon_\theta(r_2,t) = (1+\nu_s)\alpha_s \Theta_s - \sigma_r(r_2) \frac{r_2}{h} \frac{1-\nu_s^2}{E_s} \qquad (1.4.25)$$

Also let

$$\int_{r_1}^{r_2} \frac{2r_2^2}{r^3} G\{\xi(r,t)-\xi(r,\tau)\}\,dr = G^*(t,\tau) \qquad (1.4.26)$$

Then (1.4.23) becomes:

$$\sigma_{r_2} = \int_{r_1}^{r_2} F(r,t)\,dr + \int_0^t G^*(t,\tau)\frac{\partial \epsilon_\theta(r_2,\tau)}{\partial \tau}\,d\tau \qquad (1.4.27)$$

Eliminating σ_{r_2} from (1.4.25) and (1.4.27) we get

$$\frac{E_s\alpha_s\Theta_s}{(1-\nu_s)r_2}h + \int_{r_1}^{r_2} F(r,t)\,dr = \frac{E_s h}{(1-\nu_s^2)r_2}\epsilon_\theta(r_2,t)$$

$$+ \int_0^t G^*(t,\tau)\frac{\partial}{\partial \tau}\epsilon_\theta(r_2,\tau)\,d\tau \qquad (1.4.28)$$

Eq. (1.4.28) is a Volterra integral equation of the second kind. The solution of this equation - see Appendix II - for $\epsilon_\theta(r_2,t)$ and substitution for $\epsilon_\theta(r_2,t)$ in (1.4.23) yields the complete solution of the problem in the zero'th approximation.

The circumferential stress σ_θ is obtainable from (1.4.8) i.e.

$$\sigma_\theta = r\frac{\partial \sigma_r}{\partial r} + \sigma_r \tag{1.4.29}$$

To determine σ we make use of the identity

$$\sigma = 3\left(\sigma_r - S_r\right) \tag{1.4.30}$$

where

$$S_r = \int_0^t G(\xi-\xi')\frac{\partial e_r}{\partial \tau}d\tau \tag{1.4.31}$$

Now

$$e_r = \epsilon_r - \frac{\epsilon}{3} = \frac{1}{3}\left\{2r\frac{\partial \epsilon_\theta}{\partial r} + \epsilon_\theta\right\} \tag{1.4.32}$$

In view of (1.4.30), (1.4.31), (1.4.32) and (1.4.23), σ_o can be determined.

Substituting for σ_o in (1.4.9) and repeating the above procedure, we can find a first approximation tothe solution in terms of some fictition temperature Θ_1 where

$$\Theta_1 = \Theta + \frac{\sigma}{9\alpha_o K}$$

A systematic formulation of the iteration process. together with

sufficient conditions for its convergence is given in Appendix II.

A computer program based on this theory has been developed 9 . by means of which, the stress and strain distributions are found in a hollow cylinder contained in a rigid shell. The results and the corresponding temperature histories are shown in Figs. (1.4.2), (1.4.3), (1.4.4), (1.4.5), and (1.4.6).

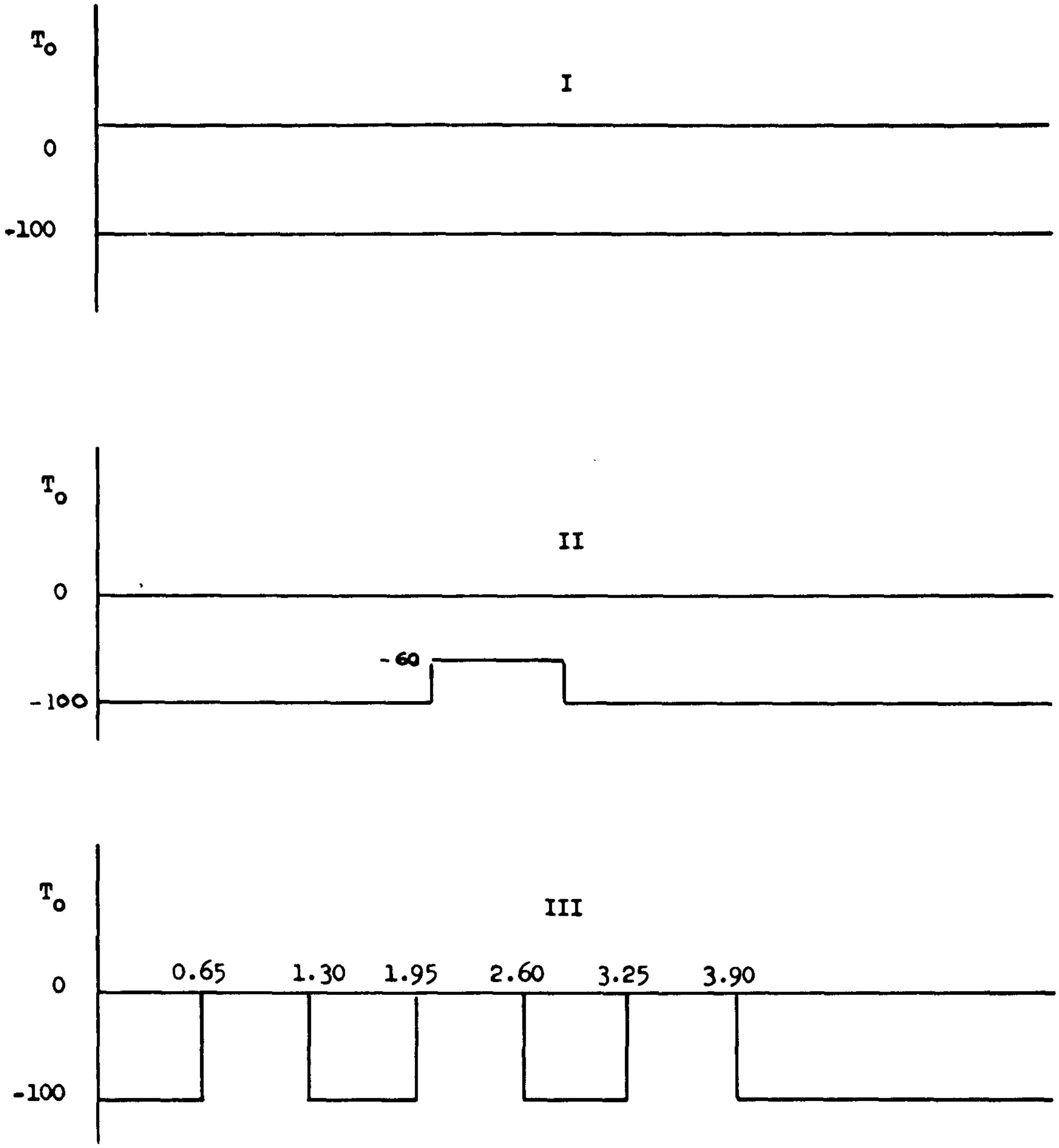


Figure (1.4.2)
CASE TEMPERATURE VARIATIONS VERSUS TIME IN HOURS

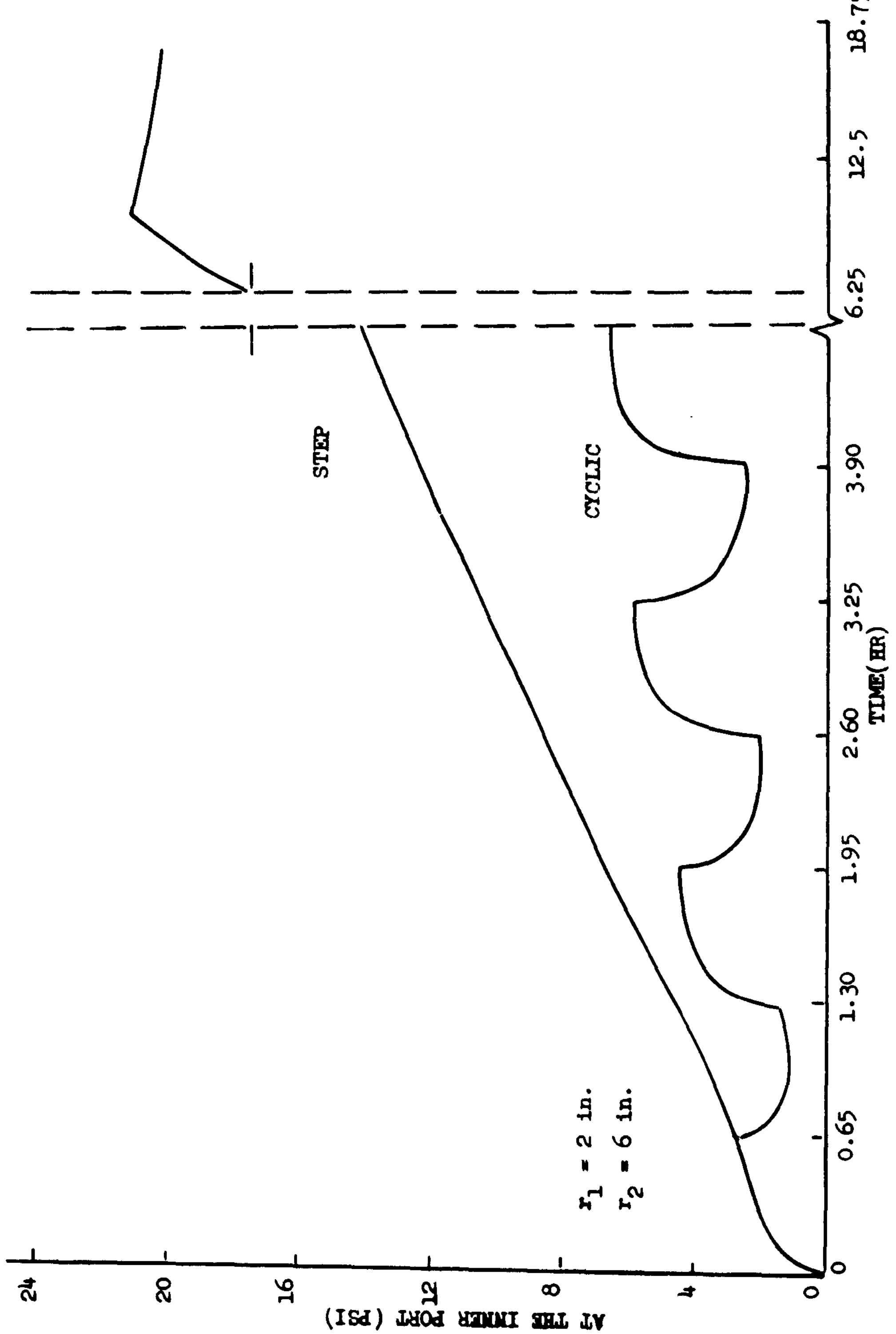


Figure (1.4.3)

HOOP STRESS VERSUS TIME FOR CYCLIC AND STEP INPUTS

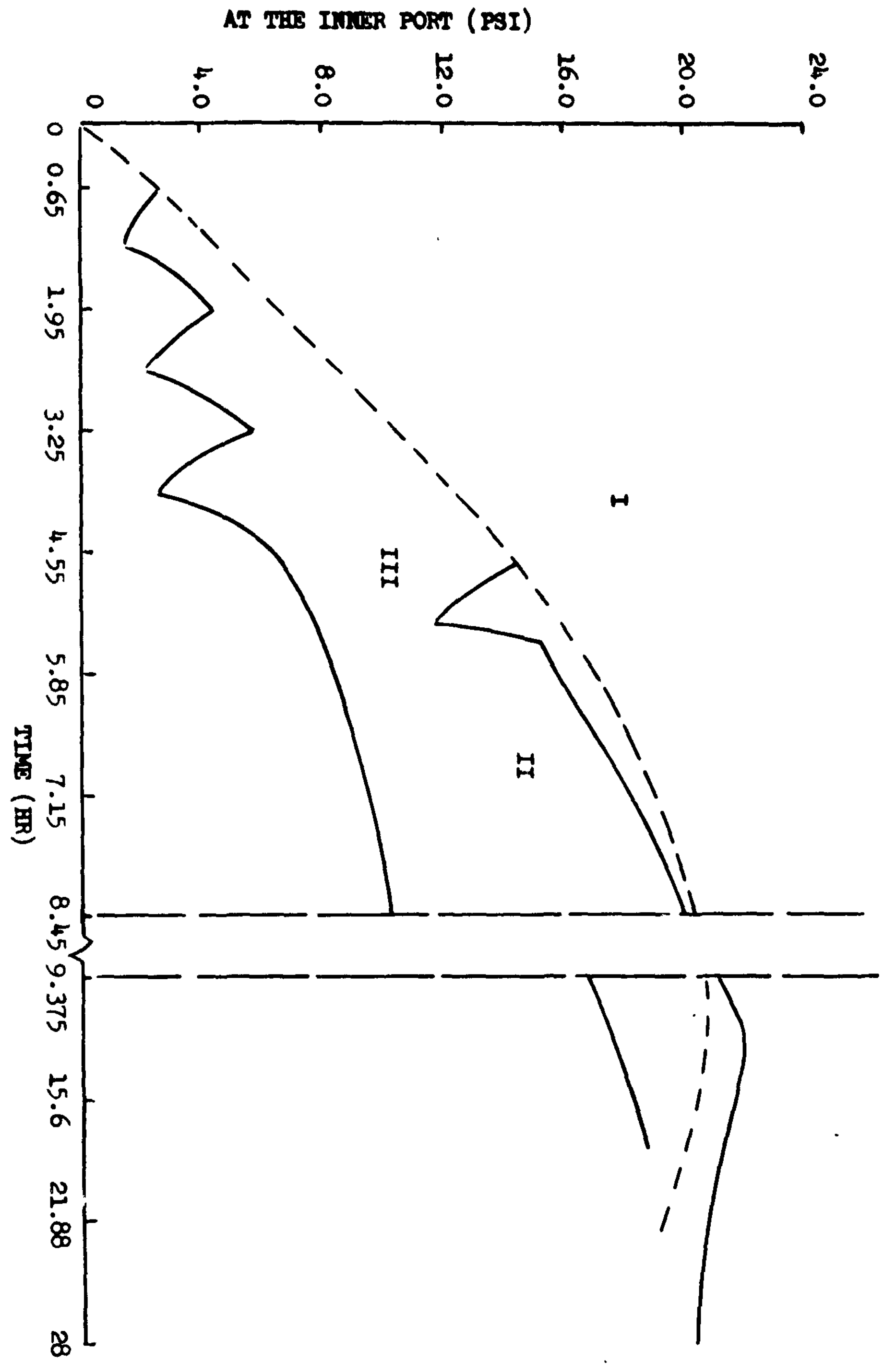


Figure (1.4.4)

HOOP STRESS VERSUS TIME FOR STEP, STEP WITH A JOG, AND CYCLIC INPUTS

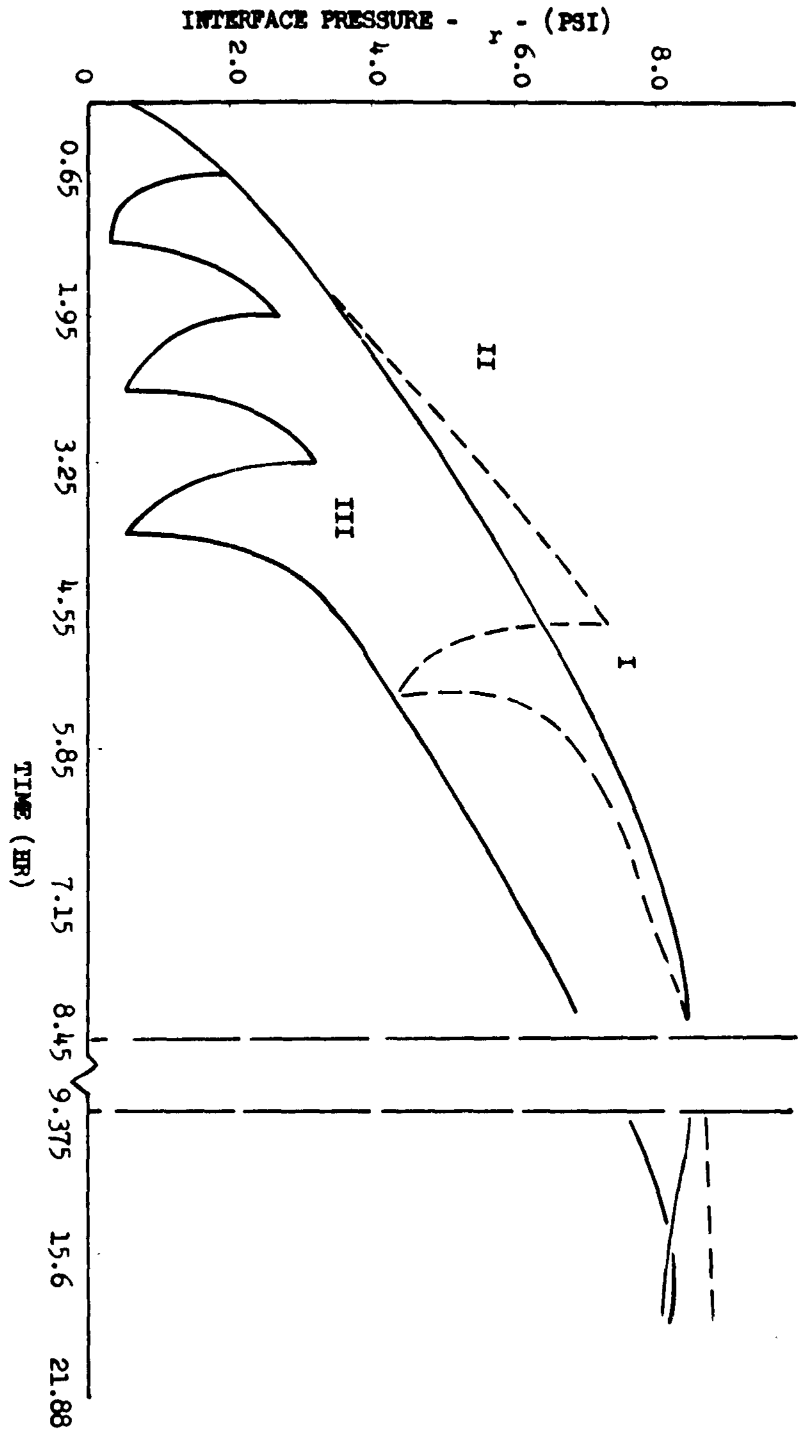


Figure (1.4.5)
INTERFACE PRESSURE VERSUS TIME FOR STEP, STEP WITH A JOG, AND CYCLIC INPUTS

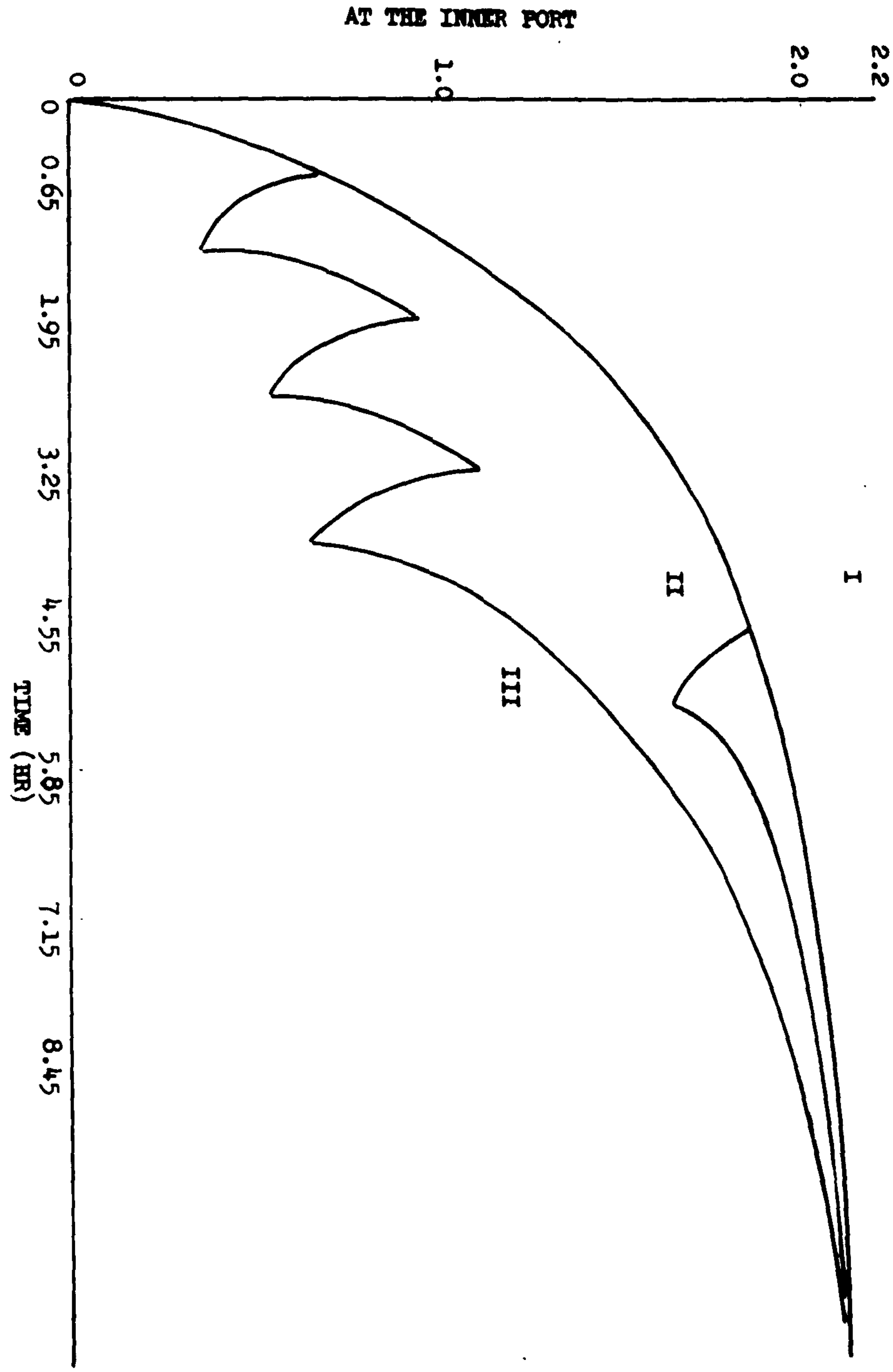


Figure (1.4.6)
INNER PORT HOOP STRAIN VERSUS TIME

1.5 A Better Zero'th Approximation to the Solution "ab initio"

By a slightly different approach a closed form solution is obtained which involves a less drastic approximation than the one employed in the previous section.

Consider the following elastic relations which can be immediately generalized to the corresponding viscoelastic relations.

$$\epsilon_z = \frac{1}{E}\left\{ \sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right\} + \alpha_0 \Theta \tag{1.5.1}$$

Since $\epsilon_z = 0$ we get

$$\sigma_z = \nu\left(\sigma_r + \sigma_\theta\right) - E\,\alpha_0\,\Theta \tag{1.5.2}$$

and hence:

$$\sigma = (1+\nu)\left(\sigma_r + \sigma_\theta\right) - E\,\alpha_0\,\Theta \tag{1.5.3}$$

However

$$(1+\nu) = \frac{3}{2}\left\{1 - \frac{E}{9K}\right\} \tag{1.5.4}$$

and the corresponding viscoelastic relation for constant K is

$$1+\nu(t) = \frac{3}{2}\left\{1 - \frac{E(t)}{9K}\right\} \tag{1.5.5}$$

Since K is constant and $E(t)$ varies between E_o and zero. for uncrossed linked, and E_o and E_R where $E_R \approx \frac{1}{3} E_o$ for a real propellant,

$$1+\nu_o \leq 1+\nu(t) \leq \frac{3}{2} \qquad (1.5.6)$$

For polmethyl methacrylate $\nu_o = .35$, which is a low value for viscoelastic materials. However even in this case

$$1.35 \leq 1+\nu(t) \leq 1.5 \qquad (1.5.7)$$

i.e. $1+\nu(t)$ varies within narrow limits and thus one is justified to make the approximation

$$1+\nu(t) \approx \frac{1.5+1.35}{2} = 1.425 = 1+\underline{\nu} \qquad (1.5.8)$$

where $\underline{\nu} = .425$ in this case.

Then by considering a constant average $\underline{\nu}$ for a viscoelastic material we will have an expression analogous to (1.5.3):

$$\sigma = (1+\underline{\nu})(\sigma_r+\sigma_\theta) - \alpha_o \int_0^t E(\xi-\xi')\frac{\partial \Theta}{\partial \tau}\, d\tau \qquad (1.5.9)$$

In view of (1.4.8) we get

$$\sigma = (1+\nu)\left(2\sigma_r + r\frac{\partial\sigma_r}{\partial r}\right) - \alpha_0 \int_0^t E(\xi-\xi')\frac{\partial \Theta}{\partial \tau}\,d\tau \qquad (1.5.10)$$

or

$$\sigma = (1+\nu)\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\sigma_r\right) - \alpha_0 \int_0^t E(\xi-\xi')\frac{\partial \Theta}{\partial \tau}\,d\tau \qquad (1.5.11)$$

also

$$\epsilon = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\epsilon_\theta\right) = \frac{\sigma}{3K} + 3\alpha_0\Theta \qquad (1.5.12)$$

Combining (1.5.11) and (1.5.12)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\epsilon_\theta\right) = \frac{1+\nu}{3K}\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\sigma_r\right) - \frac{\alpha_0}{3K}\int_0^t E(\xi-\xi')\frac{\partial \Theta}{\partial \tau}\,d\tau + 3\alpha_0\Theta \qquad (1.5.13)$$

Integrating (1.5.13)

$$r^2\epsilon_\theta = \frac{1+\nu}{3K}r^2\sigma_r - \int_{r_2}^{r}\frac{\alpha_0}{3K}\rho\int_0^t E(\xi-\xi')\frac{\partial \Theta}{\partial \tau}\,d\tau\,d\rho$$

$$+ 3\alpha_0\int_{r_2}^{r}\rho\,\Theta\,d\rho + C_1(t) \qquad (1.5.14)$$

and

$$\epsilon_\theta = \frac{1+\nu}{3K}\sigma_r - \frac{1}{r^2}\int_{r_2}^{r}\frac{\alpha_0}{3K}\rho\int_0^t E(\xi-\xi')\frac{\partial\Theta}{\partial\tau}d\tau\, d\rho$$

$$+\frac{3\alpha_0}{r^2}\int_{r_2}^{r}\rho\,\Theta\, d\rho + \frac{C_1(t)}{r^2} \qquad (1.5.15)$$

Let

$$\frac{1}{r^2}\int_{r_2}^{r}\rho\int_0^t E(\xi-\xi')\frac{\partial\Theta}{\partial\tau}d\tau\, d\rho = \Omega(r,t) \qquad (1.5.16)$$

In view of (1.5.16) and (1.4.15) we get:

$$\epsilon_\theta = \frac{1+\nu}{3K}\sigma_r - \frac{\alpha_0}{3K}\Omega + 3\alpha_0\Psi + \frac{C_1(t)}{r^2} \qquad (1.5.17)$$

where $$\Omega(r_2) = \Psi(r_2) = 0 \qquad (1.5.18)$$

We now differentiate (1.5.17) with respect to r and denote such differentiation by a dot, then after differentiation with respect to time, multiplication by $G(\xi-\xi')$, and integration between the limits of 0 and t we get:

$$\int_0^t G(\xi-\xi')\frac{\partial\dot{\epsilon}_\theta}{\partial\tau}d\tau = \frac{1+\nu}{3K}\int_0^t G(\xi-\xi')\frac{\partial\dot{\sigma}_r}{\partial\tau}d\tau$$

$$-\frac{\alpha_0}{3K}\int_0^t G(\xi-\xi')\frac{\partial\dot{\Omega}}{\partial\tau}d\tau + 3\alpha_0\int_0^t G(\xi-\xi')\frac{\partial\dot{\psi}}{\partial\tau}d\tau$$

$$-\frac{2}{r^3}\int_0^t G(\xi-\xi')\frac{\partial C_1}{\partial\tau}d\tau =$$

$$= \sigma_r \tag{1.5.19}$$

by (1.4.19).

Let us consider now the functions:

$$\xi = \xi(r,t) \quad , \quad \xi' = \xi(r,\tau) \tag{1.5.20}$$

For fixed r , ξ is a monotonic function of t and hence it can be inverted in the form

$$t = g(r,\xi) \tag{1.5.21}$$

Then if f is a function of r and t ,

$$f = f\{r, g(r,\xi)\} = \hat{f}(r,\xi) \tag{1.5.22}$$

Also let

$$\frac{\partial \sigma_r}{\partial r} \equiv \Sigma(r,t) = \hat{\Sigma}(r,\xi)$$
$$\frac{\partial \Omega}{\partial r} \equiv \Phi(r,t) = \hat{\Phi}(r,\xi) \qquad (1.5.23)$$
$$\frac{\partial \psi}{\partial r} \equiv \Pi(r,t) = \hat{\Pi}(r,\xi)$$

Then substituting (1.5.23) in (1.5.19) we obtain

$$-\Sigma(r,t) = \frac{1+\nu}{3K}\int_0^t G(\xi-\xi')\frac{\partial \Sigma}{\partial \tau}\,d\tau - \frac{\alpha_0}{3K}\int_0^t G(\xi-\xi')\frac{\partial \Phi}{\partial \tau}\,d\tau$$

$$+ 3\alpha_0\int_0^t G(\xi-\xi')\frac{\partial \Pi}{\partial \tau}\,d\tau - \frac{2}{r^3}\int_0^t G(\xi-\xi')\frac{\partial C_1}{\partial \tau}\,d\tau \qquad (1.5.24)$$

Referring all quantities to the r, ξ plane (1.5.24) becomes

$$-\hat{\Sigma}(r,\xi) = \frac{1+\nu}{3K}\int_0^\xi G(\xi-\xi')\frac{\partial \hat{\Sigma}}{\partial \xi'}\,d\xi' - \frac{\alpha_0}{3K}\int_0^\xi G(\xi-\xi')\frac{\partial \hat{\Phi}}{\partial \xi'}\,d\xi'$$

$$+ 3\alpha_0\int_0^\xi G(\xi-\xi')\frac{\partial \hat{\Pi}}{\partial \xi'}\,d\xi' - \frac{2}{r^3}\int_0^\xi G(\xi-\xi')\frac{\partial \hat{C}_1}{\partial \xi'}\,d\xi' \qquad (1.5.25)$$

We now take Laplace transform of (1.5.25) with respect to ξ and obtain:

$$-\bar{\hat{\Sigma}} = \frac{1+\nu}{3K} p\bar{G}\,\bar{\hat{\Sigma}} - \frac{\alpha_0}{3K} p\,\bar{G}\,\bar{\hat{\Phi}}$$

$$+3\alpha_0 p\bar{G}\,\bar{\hat{\Pi}} - \frac{2}{r^3}\bar{G}\,p\,\bar{\hat{C}}_1 \qquad (1.5.26)$$

Now letting

$$\frac{\bar{G}}{1+\frac{1+\nu}{3K}p\bar{G}} = \bar{R}(p) \qquad (1.5.27)$$

(1.5.26) becomes:

$$\bar{\hat{\Sigma}} = \frac{\alpha_0}{3K}\bar{R}\,p\,\bar{\hat{\Phi}} - 3\alpha_0 p\bar{R}\,\bar{\hat{\Pi}} + \frac{2}{r^3}p\bar{R}\,\bar{\hat{C}}_1 \qquad (1.5.28)$$

Now taking the inverse Laplace transform we find

$$\hat{\Sigma}(r,\xi) = \frac{\alpha_0}{3K}\int_0^{\xi} R(\xi-\xi')\frac{\partial\hat{\Phi}}{\partial\xi'}d\xi' - 3\alpha_0\int_0^{\xi} R(\xi-\xi')\frac{\partial\hat{\Pi}}{\partial\xi'}d\xi'$$

$$+\frac{2}{r^3}\int_0^{\xi} R(\xi-\xi')\frac{\partial\hat{C}_1}{\partial\xi'}d\xi' \qquad (1.5.29)$$

Finally reverting to the r, t plane (1.5.29) takes the form.

$$\Sigma(r,t) = \frac{\alpha_0}{3K}\int_0^t R(\xi-\xi')\frac{\partial \phi}{\partial \tau}\,d\tau - 3\alpha_0\int_0^t R(\xi-\xi')\frac{\partial \Pi}{\partial \tau}\,d\tau$$

$$+\frac{2}{r^3}\int_0^t R(\xi-\xi')\frac{\partial C_1}{\partial \tau}\,d\tau \tag{1.5.30}$$

Substituting for Σ, Φ, Π from (1.5.23) and after putting

$$\int_0^t R(\xi-\xi')\frac{\partial}{\partial \tau}\left\{\frac{\alpha_0\dot{\Omega}}{3K} - 3\alpha_0\dot{\psi}\right\}d\tau = \chi(r,t) \tag{1.5.31}$$

we obtain

$$\frac{\partial \sigma_r}{\partial r} = \chi(r,t) + \frac{2}{r^3}\int_0^t R(\xi-\xi')\frac{\partial C_1}{\partial \tau}\,d\tau \tag{1.5.32}$$

Hence:

$$\sigma_r = \int_{r_1}^{r}\chi(\rho,t)\,d\rho + \int_{r_1}^{r}\frac{2}{\rho^3}\int_0^t R(\xi-\xi')\frac{\partial C_1}{\partial \tau}\,d\tau\,d\rho \tag{1.5.33}$$

since $\sigma_r(r_1)=0$, $r=r_1$ being a free surface. Therefore,

$$\sigma_r(r_2,t) = \int_{r_1}^{r_2} \chi(r,t)\,dr + \int_{r_1}^{r_2} \frac{2}{r^3} \int_0^t R(\xi-\xi')\frac{\partial C_1}{\partial \tau}\,d\tau \qquad (1.5.34)$$

Also in view of (1.5.17) and (1.5.18),

$$\epsilon_{\theta_2} = \frac{1+\nu}{3K}\,\sigma_{r_2} + \frac{C_1(t)}{r_2^2} \qquad (1.5.35)$$

$$\epsilon_{\theta_2} = \epsilon_\theta(r_2,t)\ ,\quad \sigma_{r_2}' = \sigma_r(r_2,t) \qquad (1.5.36)$$

At the interface of the viscoelastic cylinder and the elastic shell we have the relation

$$\epsilon_{\theta_2} = (1+\nu_s)\alpha_s\,\Theta_s - \sigma_{r_2}\frac{r_2}{h}\,\frac{1-\nu_s^2}{E_s} \qquad (1.5.37)$$

where suffix s refers to the shell and Θ_s is the mean temperature over the shell thickness.

Eliminating ϵ_{θ_2} from (1.5.35) and (1.5.37) we obtain:

$$\frac{1+\nu}{3K}\,\sigma_{r_2} + \frac{C_1(t)}{r_2^2} = (1+\nu_s)\,\alpha_s\,\Theta_s - \sigma_{r_2}\,\frac{1-\nu_s^2}{E_s}\left(\frac{r_2}{h}\right) \qquad (1.5.38)$$

In view of (1.5.34) and (1.5.38)

$$r_2^{-2}\left\{\frac{1+\nu}{3K}+\frac{1-\nu_s^2}{E_s}\frac{r_2}{h}\right\}^{-1} C_1(t)+\int_0^t R^*(t,\tau)\frac{\partial C_1}{\partial \tau}\,d\tau$$

$$+(1+\nu_s)\alpha_s \Theta_s\left\{\frac{1+\nu}{3K}+\frac{1-\nu_s^2}{E_s}\frac{r_2}{h}\right\}^{-1}-\int_{r_1}^{r_2}\chi(r,t)\,dr \qquad (1.5.39)$$

where

$$R^*(t,\tau)=\int_{r_1}^{r_2}\frac{1}{r^3}\,R\{\xi(r,t)-\xi(r,\tau)\}\,dr \qquad (1.5.40)$$

Eq. (1.5.39) is again a Volterra Integral equation of the second kind in $C_1(t)$.

Solution of this equation and substitution in (1.5.33) solves the problem completely.

The hoop stress can be found immediately from,

$$\sigma_\theta=\sigma_r+r\frac{\partial\sigma_r}{\partial r} \qquad (1.5.41)$$

Since this solution is dependent on the aforementioned approximation one can test its accuracy by calculating σ from the expression,

$$\sigma = \sigma_r + \sigma_\theta + \int_0^t \nu(\xi-\xi')\frac{\partial}{\partial\tau}(\sigma_r+\sigma_\theta)d\tau - \alpha_0\int_0^t E(\xi-\xi')\frac{\partial\Theta}{\partial\tau}d\tau \quad (1.5.42)$$

and then use the iterative method developed in this chapter to calculate a more exact value of σ_r . However, comparison of σ as obtained from (1.5.42) as opposed to σ obtained from the approximate expression (1.5.9) will determine whether an iteration should be necessary.

CHAPTER II

Dynamic Stresses In Thermorheologically Simple Viscoelastic Bodies

2.1 Introduction

Dynamic stresses in linear viscoelastic elastic solids under non-isothermal conditions are still an unexplored field. Hopes of obtaining closed form solutions even for the simplest configurations are rather small as will be appreciated from the contents of this report.

One will recall that the quasistatic problem of the slab and the sphere were given a closed form solution, formally at least because the relevant equilibrium equation, in both cases, could be integrated directly. When, however inertia forces are taken into account this is no longer possible.

In the present chapter progress has been made by limiting our attention to incompressible viscoelastic materials, in the sense that volumetric changes either due to mechanical forces or temperature fields are zero.

Consequently the dynamic stresses examined here, arise because of the time wise variation of the mechanical forces (stresses) applied at the boundary.

We limit ourselves to the configurations of the sphere and the infinite hollow cylinder, both with polar symmetry, so that dependent variables are functions of the radius only and time. On the other hand, within this restriction, temperature fields are both non-homogeneous and transient in nature.

The solutions of both problems reduce to Volterra integral equations of the second kind, which can be systematically solved numerically without undue difficulty.

In the first part of the report we treat the viscoelastic sphere with polar symmetry. In the second part we treat the hollow viscoelastic cylinder infinite in length and also with polar symmetry. In both cases the dynamic stresses are examined in the presence of time-varying mechanical forces

2.2 Dynamic Stresses in a Viscoelastic Sphere

Sphere With Infinite External Radius

We first investigate the case of a sphere with an infinite external radius, so that, essentially, we have a viscoelastic continuum with a spherical cavity, in a transient inhomogeneous temperature field with polar symmetry. The cavity is subjected to mechanical pressure which may vary with time. It is assumed that prior to time $t=0$ the pressure was constant and that any variation began at time $t = 0+$.

The temperature field, however, may have existed prior to the application of the pressure, the time of application of the temperature field, relative to that of the applied pressure variation, being arbitrary.

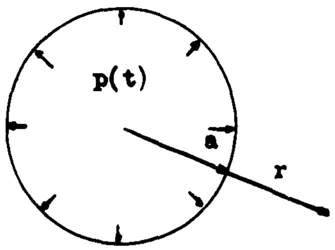


Figure (2.2.1)

Let $p(t)$, be the pressure in the cavity, "a" the radius of the cavity, r the radial distance from the center of the cavity, and u the displacement along r .

Then in the usual notation, and in view of the assumed polar symmetry we have,

$$\epsilon_r = \frac{\partial u}{\partial r} \quad , \quad \epsilon_\theta = \epsilon_\phi = \frac{u}{r} \tag{2.2.1}$$

$$\epsilon = \epsilon_{KK} = \frac{\partial u}{\partial r} + \frac{2u}{r} \quad , \quad K = \infty \tag{2.2.2}$$

$$\sigma_r - \sigma_\theta = \int_0^t G(\xi - \xi') \frac{\partial}{\partial \tau} \left\{ \epsilon_r - \epsilon_\theta \right\} d\tau \tag{2.2.3}$$

$$\epsilon = \frac{\sigma}{K} = 0 \tag{2.2.4}$$

The equilibrium relation is

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = \rho \frac{\partial^2 u}{\partial t^2} \tag{2.2.5}$$

From (2.1.4) and (2.2.2) we get

$$\frac{\partial u}{\partial r} + \frac{2u}{r} = 0 \tag{2.2.6}$$

Hence

$$u = C(t)\, r^{-2} \tag{2.2.7}$$

where $C(t)$ is a constant of integration.

Also in view of (2.2.1)

$$\left.\begin{aligned} \epsilon_r &= -2\, C(t)\, r^{-3} \\ \epsilon_\theta &= C(t)\, r^{-3} \end{aligned}\right\} \tag{2.2.8}$$

$$\epsilon_r - \epsilon_\theta = -3 r^{-3} C(t) \tag{2.2.9}$$

Hence from (2.2.3)

$$\sigma_r - \sigma_\theta = -3 r^{-3} \int_0^t G(\xi - \xi') \frac{\partial C}{\partial \tau}\, d\tau \tag{2.2.10}$$

Substituting in (2.2.5) we get

$$\frac{\partial \sigma_r}{\partial r} = 6 r^{-4} \int_0^t G(\xi - \xi') \frac{\partial C}{\partial \tau}\, d\tau + \frac{\rho}{r^2} \frac{\partial^2 C}{\partial t^2} \tag{2.2.11}$$

Integrating (2.2.11) and in view of the fact that $\sigma_r(a) = -p(t)$ we get

$$\sigma_r + p(t) = 6 \int_a^r r'^{-4} \left\{ \int_0^t G(\xi - \xi') \frac{\partial C}{\partial \tau}\, d\tau \right\} dr' + \frac{\rho}{a}\left(1 - \frac{a}{r}\right) \frac{\partial^2 C}{\partial t^2} \tag{2.2.12}$$

Interchanging the order of integration in (2.2.12)

$$\sigma_r + p(t) = 6 \int_0^t \frac{\partial C}{\partial \tau} \int_a^r r'^{-4} G(\xi - \xi')\, dr'\, d\tau + \frac{\rho}{a}\left(1 - \frac{a}{r}\right) \frac{\partial^2 C}{\partial t^2} \tag{2.2.13}$$

Now
$$\int_a^r r'^{-4} G(\xi-\xi')\,dr' = \int_a^\infty r'^{-4} G(\xi-\xi')\,dr' - \int_r^\infty r'^{-4} G(\xi-\xi')\,dr' \qquad (2.2.14)$$

provided that the improper integrals exist, and they do, since G is bounded and r'^{-4} is amonotonically decreasing function* Let,

$$\int_a^\infty r'^{-4} G\{\xi(r',t)-\xi(r',\tau)\}\,dr' = K(t,\tau) \qquad (2.2.15)$$

and

$$\int_r^\infty r'^{-4} G\{\xi(r',t)-\xi(r',\tau)\}\,dr' = N(t,\tau,r) \qquad (2.2.16)$$

Substitute (2.2.14), (2.2.15) and (2.2.16) in (2.2.13) we get

$$\sigma_r' + p(t) = \sigma\int_0^t \frac{\partial c}{\partial \tau}\{K(t,\tau) - N(t,\tau,r)\}\,d\tau + \frac{\rho}{a}\left(1-\frac{a}{r}\right)\frac{\partial^2 c}{\partial t^2} \qquad (2.2.17)$$

Note that

$$N(t,\tau,r) \to 0 \qquad (2.2.18)$$

as $r \to \infty$.

Also since $\sigma_r \to 0$ as $r \to \infty$, (2.2.17) becomes

---

* At rate greater than $\frac{1}{r}$.

$$p(t) = \sigma \int_0^t K(t,\tau) \frac{\partial c}{\partial \tau} d\tau + \frac{\rho}{a} \frac{\partial^2 c}{\partial t^2} \tag{2.2.19}$$

Eq. (2.2.19) completely determines $C(t)$, hence, formally at least, the problem is solved.

Also in view of (2.2.19), eq. (2.2.17) becomes

$$\sigma_r = -\sigma \int_0^t N(t,\tau,r) \frac{\partial c}{\partial \tau} d\tau - \frac{\rho}{a} \frac{\partial^2 c}{\partial t^2} \tag{2.2.20}$$

or

$$\sigma_r = -\sigma \int_0^t N(t,\tau,r) \frac{\partial c}{\partial \tau} d\tau - \left(\frac{a}{r}\right) p(t)$$

$$- \left(\frac{a}{r}\right) \int_0^t K(t,\tau) \frac{\partial c}{\partial \tau} d\tau =$$

$$= -\sigma \int_0^t \left\{ N(t,\tau,r) + \frac{a}{r} K(t,\tau) \right\} \frac{\partial c}{\partial \tau} d\tau - \frac{a}{r} p(t) \tag{2.2.21}$$

Hence the solution to the problem hinges on the solution of (2.2.19).

Let

$$\frac{\partial^2 c}{\partial t^2} = g(t) \tag{2.2.22}$$

Then

$$\frac{\partial c}{\partial t} = \int_0^t g(\tau) d\tau \tag{2.2.23}$$

since from (2.2.7), by virtue of $u=0, \frac{\partial u}{\partial t}=0,$ at $t = 0+$

$$C\Big|_{t=0+} = 0 \quad , \quad \frac{\partial C}{\partial t}\Big|_{t=0+} = 0 \tag{2.2.24}$$

Then

$$\int_0^t K(t,\tau)\frac{\partial C}{\partial \tau}\,d\tau = \int_0^t K(t,\tau)\int_0^\tau g(t')\,dt'\,d\tau \tag{2.2.25}$$

However regarding (2.2.25) as a double integral we get

$$\int_0^t\int_0^\tau K(t,\tau)\,g(t')\,dt'\,d\tau =$$

$$= \int_0^t g(t')\left\{\int_{t'}^t K(t,\tau)\,d\tau\right\}dt' \tag{2.2.26}$$

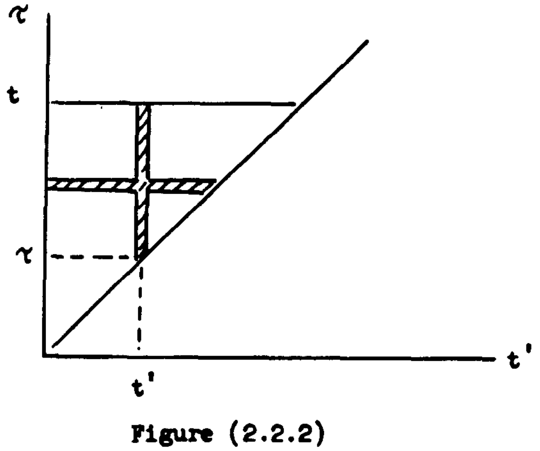


Figure (2.2.2)

Let

$$\int_{t'}^t K(t,\tau)\,d\tau = K^*(t,t') \tag{2.2.27}$$

Then

$$\int_0^t K(t,\tau)\frac{\partial c}{\partial \tau}\,d\tau = \int_0^t \int_0^\tau K(t,\tau)\,q(t')\,dt'\,d\tau$$
$$= \int_0^t K^*(t,\tau)\,q(\tau)\,d\tau \qquad (2.2.28)$$

Substituting (2.2.22) and (2.2.28) in (2.2.19) we get

$$\frac{a}{\rho}\,p(t) = \frac{6a}{\rho}\int_0^t K^*(t,\tau)\,q(\tau)\,d\tau + q(t) \qquad (2.2.29)$$

where

$$K^*(t,\tau) = \int_\tau^t K(t,t')\,dt' \qquad (2.2.29a)$$

Equation (2.2.29) is a classical Volterra Integral Equation of the second kind which can be solved numerically by various means.

2.3 Temperature Uniform But Time-Dependent

In this case eq. (2.2.11) can be directly integrated with respect to r , to give,

$$\sigma_r + p(t) = \frac{2}{a^3}\int_0^t G(\xi-\xi')\,\frac{\partial c}{\partial \tau}\,d\tau \cdot \left\{1-\left(\frac{a}{r}\right)^3\right\} + \frac{\rho}{a}\left(1-\frac{a}{r}\right)\frac{\partial^2 c}{\partial t^2} \qquad (2.3.1)$$

Since $\sigma_r \to 0$ as $r \to \infty$ we get

$$p(t) = \frac{2}{a^3}\int_0^t G(\xi-\xi')\,\frac{\partial c}{\partial \tau}\,d\tau + \frac{\rho}{a}\,\frac{\partial^2 c}{\partial t^2} \qquad (2.3.2)$$

or

$$\frac{d^2c}{dt^2} + \frac{2}{a^2\rho}\int_0^t G(\xi-\xi')\frac{dc}{d\tau}\,d\tau = \frac{a}{\rho}\,p(t) \tag{2.3.3}$$

Let

$$K^*(t,\tau) = \frac{1}{G_0}\int_\tau^t G\{\xi(t)-\xi(\tau')\}\,d\tau' \tag{2.3.4}$$

$$\frac{2G_0}{a^2\rho} = \lambda \tag{2.3.5}$$

where $G_o = G(o)$.

Then (2.3.3) becomes

$$g(t) + \lambda\int_0^t K^*(t,\tau)\,g(\tau)\,d\tau = \frac{a}{\rho}\,p(t) \tag{2.3.6}$$

Also from (2.3.1) and (2.2.22)

$$\sigma_r = -\frac{2}{r^3}\int_0^t G(\xi-\xi')\frac{dc}{d\tau}\,d\tau - \frac{\rho}{r}\,g(t) \tag{2.3.7}$$

By virtue of (2.3.2) and (2.3.7)

$$\sigma_r = -\left(\frac{a}{r}\right)^3 p(t) + \frac{\rho}{r}\left\{\left(\frac{a}{r}\right)^2 - 1\right\} g(t) \tag{2.3.8}$$

where g(t) is given from (2.3.6).

From (2.2.5) and (2.2.7)

$$2\sigma_\theta = r\frac{d\sigma_r}{dr} + 2\sigma_r - (\rho/r)\,g(t)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r^2\sigma_r) - (\rho/r)\,g(t) = \left(\frac{a}{r}\right)^3 p(t) - \frac{\rho a^2}{r^3}\,g(t) - \frac{2\rho}{r}\,g(t)$$

$$= \left(\frac{a}{r}\right)^3 p(t) - \frac{\rho}{r}\,g(t)\left[\,2 + \left(\frac{a}{r}\right)^2\right] \qquad (2.3.9)$$

2.4 Sphere With Finite External Radius

a. Condition of Fixity at External Radius. $r = b$

If u is to vanish at r = b, $c(t) \equiv 0$ from (2.2.7).

Hence $u \equiv 0$ and all deviatoric strains and stresses vanish identically,

The equilibrium relation (2.2.5) becomes

$$\frac{\partial\sigma_r}{\partial r} = 0 \qquad (2.4.1)$$

or

$$\sigma_r = -\,p(t) \qquad (2.4.2)$$

in view of the boundary condition at r = a.

Also since

$$\sigma_r - \sigma_\theta = 0 \qquad (2.4.3)$$

$$\sigma_\theta = \sigma_\phi = -\,p(t) \qquad (2.4.4)$$

Stresses are transmitted instaneously and there are no inertia effects.

2.5 Free Surface at $r = b$

Again we quote (2.2.3), i.e.

$$\sigma_r + p(t) = G\int_0^t \frac{dc}{d\tau}\int_a^r r'^{-4} G(\xi-\xi')\,dr'\,d\tau + \frac{\rho}{a}\left(1-\frac{a}{r}\right)\frac{d^2c}{dt^2} \qquad (2.5.1)$$

at $r = b$, $\sigma_r = 0$ hence

$$p(t) = G\int_0^t \frac{dc}{d\tau}\int_a^b r'^{-4}\left\{G(\xi-\xi')\right\}dr'\,d\tau$$

$$+\frac{\rho}{a}\left(1-\frac{a}{r}\right)\frac{d^2c}{dt^2} \qquad (2.5.2)$$

Let

$$\int_a^b G(\xi-\xi')\,r'^{-4}\,dr' = L(t,\tau) \qquad (2.5.3)$$

Then (2.5.2) becomes

$$p(t) = G\int_0^t L(t,\tau)\frac{dc}{d\tau}\,d\tau + \frac{\rho}{a}\left(1-\frac{a}{b}\right)\frac{d^2c}{dt^2} \qquad (2.5.4)$$

(2.5.4) is an equation similar to (2.2.19).

On substitution of (2.5.4) in (2.5.1) we find for σ_r ,

$$\sigma_r = -G\int_0^t \frac{dc}{d\tau}\left\{\int_r^b r'^{-4} G(\xi-\xi')\,dr'\right\}d\tau$$

$$-\frac{\rho}{r}\frac{d^2c}{dt^2} \qquad (2.5.5)$$

2.6 Isothermal Sphere With Infinite Radius

From equation (2.3.3)

$$\frac{d^2C}{dt^2} + \frac{2}{a^2\rho}\int_0^t G(t-\tau)\frac{dC}{d\tau}\,d\tau = \frac{a}{\rho}\,p(t) \qquad (2.6.1)$$

Consider G(t) to be given by a Maxwell model, i.e.

$$G(t) = G_o\, e^{-\frac{t}{\lambda}} \qquad (2.6.2)$$

Taking Laplace transform of (2.6.1) and in view of (2.2.24)

$$s^2\bar{C} + \frac{2}{a^2\rho}\,\bar{G}\,s\,\bar{C} = \frac{a}{\rho}\,\bar{p}(s). \qquad (2.6.3)$$

From (2.6.2)

$$\bar{G}(s) = G_o\,\frac{1}{s+\frac{1}{\lambda}} \qquad (2.6.4)$$

substituting in (2.6.3) we obtain

$$s^2\bar{C} + \frac{2G_o}{a^2\rho}\,\frac{s\bar{C}}{s+\frac{1}{\lambda}} = \frac{a}{\rho}\,\bar{p}(s) \qquad (2.6.5)$$

Let $s^2\bar{C} = \bar{q}(s)$ and $\frac{2G_o}{a^2\rho} = \omega^2$ (2.6.6)

then (2.6.5) becomes

$$\bar{g}(s) + \frac{\omega^2 \bar{g}(s)}{s\,(s+\frac{1}{\lambda})} = \frac{a}{\rho}\,\bar{p}(s) \tag{2.6.7}$$

or

$$\bar{g}(s) = \frac{(s^2+\frac{s}{\lambda})\,\frac{a}{\rho}\,\bar{p}(s)}{s^2+\frac{s}{\lambda}+\omega^2} \tag{2.6.8}$$

or

$$\bar{g}(s) = \frac{a}{\rho}\,\bar{p}(s) - \frac{\frac{a}{\rho}\,\omega^2}{s^2+\frac{s}{\lambda}+\omega^2}\,\bar{p}(s) \tag{2.6.9}$$

Substituting (2.6.9) in (2.3.8) we find

$$-\sigma_r = -\left(\frac{a}{r}\right)^3 p(t) + \frac{\rho}{r}\left\{\left(\frac{a}{r}\right)^2 - 1\right\}\left\{\frac{a}{\rho}\,p(t) - \mathcal{L}^{-1}\frac{\omega^2\frac{a}{\rho}}{s^2+\frac{s}{\lambda}+\omega^2}\,\bar{p}(s)\right.$$

$$= -\left(\frac{a}{r}\right)p(t) + \left(\frac{a}{r}\right)\left\{1-\left(\frac{a}{r}\right)^2\right\}\mathcal{L}^{-1}\left\{\frac{\omega^2}{s^2+\frac{s}{\lambda}+\omega^2}\,\bar{p}(s)\right\} \tag{2.6.10}$$

Now

$$\mathcal{L}^{-1}\frac{\omega^2}{s^2+\frac{s}{\lambda}+\omega^2} = \frac{\omega^2}{\sqrt{\omega^2-\frac{1}{4\lambda^2}}}\,e^{-\frac{t}{2\lambda}}\sin\left(\omega^2-\frac{1}{4\lambda^2}\right)^{1/2}t = \psi(t) \tag{2.6.11}$$

Then (2.6.10) becomes

$$\sigma_r = -\left(\frac{a}{r}\right) p(t) + \frac{a}{r}\left\{1 - \left(\frac{a}{r}\right)^2\right\} \int_0^t \psi(t-\tau)\, p(\tau)\, d\tau \qquad (2.6.12)$$

which is the complete solution for all load cycles

Nature of $\Psi(t)$

(a) $\frac{1}{2\lambda} < \omega$

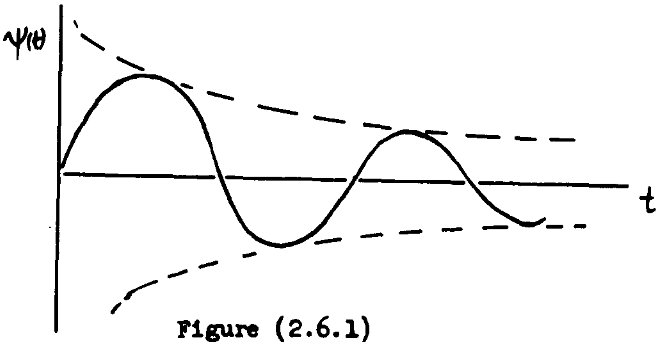


Figure (2.6.1)

(b) $\frac{1}{2\lambda} = \omega$

$$\psi(t) = \omega^2 t\, e^{-\frac{t}{2\lambda}} \qquad (2.6.13)$$

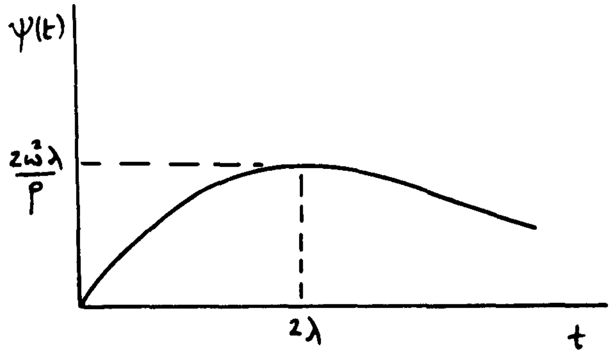


Figure (2.6.2)

(c) $\frac{1}{2\lambda} > \omega$

Call $\sqrt{\frac{1}{4\lambda^2} - \omega^2} = \omega'$, $\therefore \quad \frac{1}{2\lambda} > \omega'$

$$\psi = \frac{\omega^2}{\omega'} e^{-\frac{t}{2\lambda}} \sinh \omega' t \tag{2.6.14}$$

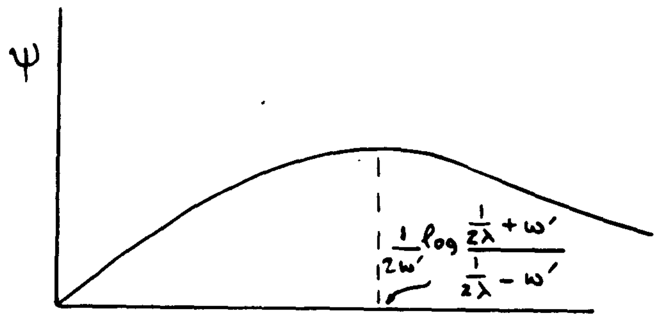


Figure (2.6.3)

2.7 Dynamic Stresses in an Infinite Hollow Cylinder

We finally consider the dynamic stresses arising in an infinite cylinder in radial transient temperature fidld. As stated in the introduction these are considered the result of varying interval pressure.

Under the above conditions, and in view of the assumed material incompressibility, we have in the usual notation:

$$\epsilon_r = \frac{\partial u}{\partial r} , \quad \epsilon_\theta = \frac{u}{r} , \quad \epsilon_z = 0 \tag{2.7.1}$$

$$\epsilon = \epsilon_r + \epsilon_\theta = \frac{\partial u}{\partial r} + \frac{u}{r} = 0 \tag{2.7.2}$$

From (2.7.2)

$$\frac{\partial}{\partial r}(ur) = 0 \qquad \therefore \qquad u = \frac{1}{r}\, C(t) \tag{2.7.3}$$

where $C(t)$ is a constant of integration.

Hence:

$$\epsilon_r = \frac{\partial u}{\partial r} = -\frac{1}{r^2}\, C(t) \tag{2.7.4}$$

$$\epsilon_\theta = \frac{1}{r^2}\, C(t) \tag{2.7.5}$$

and

$$\epsilon_r - \epsilon_\theta = -\frac{2}{r^2}\, C(t) \tag{2.7.6}$$

The equilibrium equation to be satisfied is

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u}{\partial t^2} \tag{2.7.7}$$

In view of the constitutive relation

$$s_{ij} = \int_0^t G(t-\tau)\frac{\partial e_{ij}}{\partial \tau}\, d\tau \tag{2.7.8}$$

and the thermorheologically simple nature of the cylinder

$$\sigma_r - \sigma_\theta = \int_0^t G(\xi - \xi')\frac{\partial}{\partial \tau}(\epsilon_r - \epsilon_\theta)\, d\tau \tag{2.7.9}$$

Substituting for $\epsilon_r - \epsilon_\theta$ from (2.7.6) in (2.7.9) we obtain

$$\sigma_r - \sigma_\theta = -\frac{2}{r^2}\int_0^t G(\xi-\xi')\frac{\partial c}{\partial \tau}\,d\tau \tag{2.7.10}$$

As a consequence of (2.7.10) and (2.7.3), (2.7.7) becomes:

$$\frac{\partial \sigma_r}{\partial r} = \frac{2}{r^3}\int_0^t G(\xi-\xi')\frac{\partial c}{\partial \tau}\,d\tau + \frac{\rho}{r}\frac{\partial^2 c}{\partial t^2} \tag{2.7.11}$$

<u>Free external surface</u>

Let the interval radius be "a", and the external radius be "b". Consider the following boundary conditions $\sigma_r = -p(t)$ at $r = a$,

$\sigma_r = o$ at $r = b$ (2.7.12)

In view of (2.7.12) integrating (2.7.11) we find

$$\sigma_r + p(t) = 2\int_0^t \frac{\partial c}{\partial \tau}\int_a^r \frac{1}{r'^3}\,G(\xi-\xi')\,dr'\,d\tau$$

$$+ \rho\log\left(\frac{r}{a}\right)\frac{\partial^2 c}{\partial t^2} \tag{2.7.13}$$

Since at $r = b$, $\sigma_r = o$, we have from (2.7.13).

$$p(t) = 2\int_0^t M(t,\tau)\frac{\partial c}{\partial \tau}\,d\tau + \rho\log\left(\frac{b}{a}\right)\frac{\partial^2 c}{\partial t^2} \tag{2.7.14}$$

This equation completely determines $\frac{\partial^2 c}{\partial t^2}$ and $\frac{\partial c}{\partial t}$, being analogous to (2.2.19), where

$$M(t, \tau) = \int_a^b \frac{1}{r^3} G(\xi - \xi') \, dr \qquad (2.7.15)$$

Also from (2.7.13):

$$\begin{aligned} \sigma_r + p(t) = {} & 2\int_0^t \frac{\partial c}{\partial \tau} \int_a^b \frac{1}{r^3} G(\xi - \xi') \, dr \, d\tau \\ & - 2\int_0^t \frac{\partial c}{\partial \tau} \int_r^b \frac{1}{r^3} G(\xi - \xi') \, dr \, d\tau \\ & + \rho \frac{\partial^2 c}{\partial t^2} \left\{ \log \frac{b}{a} + \log \frac{r}{b} \right\} \end{aligned} \qquad (2.7.16)$$

Incorporating (2.7.14) and (2.7.15) in (2.7.16) we obtain

$$\sigma_r = -2 \int_0^t Q(t, \tau, r) \frac{\partial c}{\partial \tau} + \rho \frac{\partial^2 c}{\partial t^2} \log \frac{r}{b} \qquad (2.7.17)$$

where

$$Q(r, t, \tau) = \int_r^b \frac{1}{r^3} G(\xi - \xi') \, dr \qquad (2.7.18)$$

<u>Cylinder contained in thin elastic shell</u>

Let the suffix s denote quantities pertaining to the shell.

At $r = b$, as a consequence of the continuity of u and σ_r at the cylinder-shell interface we have:

$$\sigma_{r_s} = -\frac{E_s}{1-\nu_s^2} \left(\frac{h}{b}\right) \frac{c(t)}{b^2} + \frac{E_s}{1-\nu_s} \alpha_{os} \Theta_s \qquad (2.7.19)$$

Incorporating this condition in (2.7.16) at $r = b$ we find:

$$\frac{E_s}{1-\nu_s}\alpha_{os}\Theta_s + p(t) = \frac{E_s}{1-\nu_s^2}\left(\frac{h}{b}\right)\frac{C(t)}{b^2} + 2\int_0^t M(t,\tau)\frac{\partial C}{\partial \tau}\,d\tau + \rho\log\left(\frac{b}{a}\right)\frac{\partial^2 C}{\partial t^2} \tag{2.7.20}$$

Eq. (2.7.20) can be reduced to the canonical form of the Volterra integral equation of the second kind by the following procedure:

Since c, $\frac{dc}{dt}$ are zero at t = o+, we have

$$C(t) = \int_0^t \int_0^\tau \frac{\partial^2 C}{\partial t^2}\,dt\,d\tau \tag{2.7.21}$$

or

$$C(t) = \int_0^t (t-\tau)\frac{\partial^2 C}{\partial \tau^2}\,d\tau \tag{2.7.22}$$

Also

$$\int_0^t M(t,\tau)\frac{\partial C}{\partial \tau}\,d\tau = \int_0^t M^*(t,\tau)\frac{\partial^2 C}{\partial \tau^2}\,d\tau \tag{2.7.23}$$

where $$M^*(t,\tau) = \int_\tau^t M(t,t')\,dt'$$

Eq. (2.7.20) in view of (2.7.21) and (2.7.23) becomes:

$$\frac{E_s}{1-\nu_s}\alpha_{os}\Theta_s + p(t) = \int_0^t \left\{2M(t,\tau) + (t-\tau)\frac{E_s}{1-\nu_s^2}\cdot\frac{h}{b}\cdot\left(\frac{1}{b}\right)^2\right\} g(\tau)\,d\tau + \rho\log\left(\frac{b}{a}\right) g(t) \tag{2.7.24}$$

where

$$g(t) = \frac{\partial^2 c}{\partial t^2} \qquad (2.7.25)$$

Calling

$$2M^{*}(t,\tau) + (t-\tau)\frac{E_s}{1-\nu_s^2}\left(\frac{h}{b}\right)\frac{1}{b^2} \equiv M^{**}(t,\tau) \qquad (2.7.26)$$

we have from (2.7.24):

$$\rho \log\left(\frac{b}{a}\right) g(t) + \int_0^t M^{**}(t,\tau)\, g(\tau)\, d\tau = p(t) + \frac{E_s}{1-\nu_s}\alpha_{os}\Theta_s \qquad (2.7.27)$$

which is the canonical form of a Volterra integral equation of the second kind.

It is worth noting that as $b \to \infty$ $\log \frac{b}{a} \to \infty$ and $g(t) \to 0$, hence $u \to 0$.

As a consequence,

$$\sigma_r = \sigma_\theta = -p(t) \qquad (2.7.28)$$

i.e. stress waves propagate instantaneously, without any subsequent vibrations being set up. A rigid external shell gives rise to the same result.

CHAPTER III

Horizontal Slump of a Viscoelastic Hollow Cylinder Contained In a Thin Elastic Shell

Synopsis

In this chapter the problem of the horizontal slump of a viscoelastic cylinder contained in a thin flexible shell resting on a rigid plane, is solved. The solution is within the scope of small deformations and classical shell theory.

The viscoelastic stress-strain law is of the integral type, and it is used in its most general form.

The solution is purely formal in as far as no numerical results are given

3.1 Introduction

The problem of the horizontal slump of a viscoelastic hollow dylinder contained in a rigid shell has been solved [15]. The solution indicated that prohibitively large displacements develop for large times. It is reasonable to expect that a flexible shell will exert a less restraining influence on the cylinder and will allow therefore even larger displacements to develop.

An exact formal solution tothe problem is obtained, within the scope of small deformations and classical shell theory, in the form of sine and cosine series in the angular coordinate.

3.2 Analysis

The treatment of isothermal viscoelastic boundary value problems is facilitated at least formally by removal of time dependence by the application of Laplace transform, and consequent reduction of the viscoelastic problem to an associated elastic one. As a first step, therefore, we obtain the solution to the problem of a hollow elastic cylinder contained in an elastic shell resting on a horizontal rigid plane, both being deformed by virtue of the forces exerted by their own weight.

We further assume that the cylinder is "long" so that plane strain in the axial direction can be assumed.

Under the above assumptions the geometry of the cylinder-shell configuration is defined by Fig. 3.4.1.

The complete solution is obtained by solving for the cylinder and shell separately, and satisfying continuity of radial and shear stresses as well as radial and circumferential displacements at the cylinder-shell interface.

3.3) Cylinder Analysis

Satisfaction of equilibrium in the radial and circumferential directional yields the following relations in the usual notation of polar coordinates;

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + \rho g \cos\theta = 0 \qquad (3.3.1)$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} - \rho g \sin\theta = 0 \qquad (3.3.2)$$

Let

$$S_r = \sigma_r - \sigma \tag{3.3.3a}$$

$$S_\theta = \sigma_\theta - \sigma \tag{3.3.3b}$$

$$\tau = \tau_{r\theta} \tag{3.3.3c}$$

where

$$\sigma = \frac{1}{3}\left\{ \sigma_r + \sigma_\theta + \sigma_z \right\} \tag{3.3.3d}$$

Then (3.3.1) and (3.3.2) become,

$$\frac{\partial S_r}{\partial r} + \frac{1}{r}\frac{\partial \tau}{\partial \theta} + \frac{S_r - S_\theta}{r} + \frac{\partial \sigma}{\partial r} + \rho g \cos\theta = 0 \tag{3.3.4}$$

$$\frac{\partial S_\theta}{\partial \theta} + r\frac{\partial \tau}{\partial r} + 2\tau + \frac{\partial \sigma}{\partial \theta} - \rho g r \sin\theta = 0 \tag{3.3.5}$$

We eliminate the dependence on θ by taking finite Fouries transform. Let

$$\bar{\tau}(r,n) = \frac{1}{\pi}\int_{-\pi}^{\pi} \tau(r,\theta)\sin n\theta \, d\theta \tag{3.3.6a}$$

$$\bar{S}_r(r,n) = \frac{1}{\pi}\int_{-\pi}^{\pi} S_r(r,\theta)\cos n\theta \, d\theta \tag{3.3.6b}$$

$$\bar{S}_\theta(r,n) = \frac{1}{\pi}\int_{-\pi}^{\pi} S_\theta(r,\theta)\cos n\theta \, d\theta \tag{3.3.6c}$$

$$\bar{\sigma}(r,n) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma(r,\theta)\cos n\theta \, d\theta \tag{3.3.6d}$$

In view of the symmetry of the problem S_r , S_θ , σ are even functions of θ but τ is an odd function. Thus we have the conditions:

$$\frac{\partial S_r}{\partial\theta}(r,0) = \frac{\partial S_r}{\partial\theta}(r,\pi) = \frac{\partial S_\theta}{\partial\theta}(r,0) = \frac{\partial S_\theta}{\partial\theta}(r,\pi) = 0 \tag{3.3.7a}$$

$$\frac{\partial\sigma}{\partial\theta}(r,0) = \frac{\partial\sigma}{\partial\theta}(r,\pi) = 0 \tag{3.3.7b}$$

$$\tau(r,0) = \tau(r,\pi) = 0 \tag{3.3.7c}$$

By virtue of (3.3.6) and (3.3.7) Eq. (3.3.4) becomes:

$$r\frac{d\bar{S}_r}{dr} + n\bar{\tau} + \bar{S}_r - \bar{S}_\theta + r\frac{d\bar{\sigma}}{dr} = \begin{cases} 0, & n \neq 1 \\ -r\rho g, & n = 1 \end{cases} \tag{3.3.8}$$

and Eq. (3.3.5) becomes

$$-n\bar{S}_\theta + r\frac{d\bar{\tau}}{dr} + 2\bar{\tau} - n\bar{\sigma} = \begin{cases} 0, & n \neq 1 \\ r\rho g, & n = 1 \end{cases} \tag{3.3.9}$$

We now assume that the material is incompressible so that σ is

indeterminate from the stress-strain relations. Hence eliminating ϕ from (3.3.8) and (3.3.9), and putting

$$\bar{S}_r - \bar{S}_\theta = \bar{\sigma}_r - \bar{\sigma}_\theta = \bar{S} \tag{3.3.10}$$

we obtain:

$$n r \frac{d\bar{S}}{dr} + n\bar{S} + n^2 \bar{\tau} + 3r\frac{d\bar{\tau}}{dr} + r^2 \frac{d^2\bar{\tau}}{dr^2} = 0 \tag{3.3.11}$$

The strain displacement relations in polar coordinate are:

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \gamma_{r\theta} = \frac{1}{2}\left\{\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right\} \tag{3.3.12}$$

$$\epsilon_z = 0, \quad \epsilon_r + \epsilon_\theta + \epsilon_z = 0 \quad \therefore \quad \epsilon_r + \epsilon_\theta = 0 \tag{3.3.13}$$

In view of the material incompressibility the stress strain relation is

$$s_{ij} = G e_{ij} \tag{3.3.14}$$

where

$$e_{ij} = \epsilon_{ij} - \frac{\epsilon_{kk}}{3} \qquad \text{(K summed)} \tag{3.3.15}$$

or putting

$$\gamma_{r\theta} = \gamma$$

$$\tau = G\gamma \tag{3.3.16}$$

$$s = (\epsilon_r - \epsilon_\theta) G = 2\epsilon_r G = -2\epsilon_\theta G \tag{3.3.17}$$

The strain compatibility relation is

$$\frac{\partial^2 \epsilon_r}{\partial \theta^2} - r\frac{\partial \epsilon_r}{\partial r} + r\frac{\partial}{\partial r^2}(r\epsilon_\theta) = 2\frac{\partial^2}{\partial r \partial \theta}(r\gamma) \tag{3.3.18}$$

or

$$\frac{\partial^2 \epsilon_r}{\partial \theta^2} - r\frac{\partial \epsilon_r}{\partial r} + r^2\frac{\partial^2 \epsilon_\theta}{\partial r^2} + 2r\frac{\partial \epsilon_\theta}{\partial r} = \\ = 2\left\{ r\frac{\partial^2 \gamma}{\partial r \partial \theta} + \frac{\partial \gamma}{\partial \theta} \right\} \tag{3.3.19}$$

Now let

$$\bar{u} = \frac{1}{\pi}\int_{-\pi}^{\pi} u(r,\theta)\cos n\theta \, d\theta \tag{3.3.20}$$

$$\bar{v} = \frac{1}{\pi}\int_{-\pi}^{\pi} v(r,\theta)\sin n\theta \, d\theta \tag{3.3.21}$$

Again in view of the existing symmetry

$$\frac{\partial u}{\partial \theta}(r,0) = \frac{\partial u}{\partial \theta}(r,\pi) = 0 \tag{3.3.22a}$$

$$v(r,0) = v(r,\pi) = 0 \tag{3.3.22b}$$

Eqs. (3.3.12) through (3.3.19) in the transformed plane become,

$$\bar{\epsilon}_r = \frac{\partial \bar{u}}{\partial r}, \quad \bar{\epsilon}_\theta = \frac{\bar{u}}{r} + \frac{n}{r}\bar{v}, \quad \bar{\gamma} = \frac{1}{2}\left\{-\frac{n}{r}\bar{u} + \frac{d\bar{v}}{dr} - \frac{\bar{v}}{r}\right\} \qquad (3.3.12a)$$

$$\bar{\epsilon}_z = 0, \quad \bar{\epsilon}_r + \bar{\epsilon}_\theta + \bar{\epsilon}_z = 0, \quad \bar{\epsilon}_r + \bar{\epsilon}_\theta = 0 \qquad (3.3.13a)$$

$$\bar{s}_{ij} = G\,\bar{e}_{ij} \qquad (3.3.14a)$$

$$\bar{e}_{ij} = \bar{\epsilon}_{ij} - \frac{1}{3}\bar{\epsilon}_{kk} \qquad (3.3.15a)$$

$$\bar{\tau} = G\,\bar{\gamma} \qquad (3.3.16a)$$

$$\bar{s} = \{\bar{\epsilon}_r - \bar{\epsilon}_\theta\}G = 2\bar{\epsilon}_r G = -2\bar{\epsilon}_\theta G \qquad (3.3.17a)$$

$$-n^2\bar{\epsilon}_r - r\frac{d\bar{\epsilon}_r}{dr} + r^2\frac{d^2\bar{\epsilon}_\theta}{dr^2} + 2r\frac{d\bar{\epsilon}_\theta}{dr} = 2\left\{nr\frac{d\bar{\gamma}}{dr} + n\bar{\gamma}\right\} \qquad (3.3.19a)$$

Substituting for $\bar{s}$ from (3.3.16) and (3.3.17) we get

$$n^2\bar{s} + 3r\frac{d\bar{s}}{dr} + r^2\frac{d^2\bar{s}}{dr^2} + 4n\left\{r\frac{d\bar{\tau}}{dr} + \bar{\tau}\right\} = 0 \qquad (3.3.23)$$

Eqs. (3.3.11) and (3.3.23) provide two ordinary linear simultaneous equations in the unknowns τ and s.

We now make a transformation in the independent variable by putting

$$\frac{r}{r_1} = x = e^{z} \qquad (3.3.24)$$

then

$$r\frac{\partial}{\partial r} = \frac{\partial}{\partial z} \quad , \quad r^2\frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z} \tag{3.3.25}$$

In view of (3.3.24) and (3.3.25) equations (3.3.11) and (3.3.20) become:

$$n\frac{d\bar{s}}{dz} + n\bar{s} + n^2\bar{\tau} + 2\frac{d\bar{\tau}}{dz} + \frac{d^2\bar{\tau}}{dz^2} = 0 \tag{3.3.26}$$

$$n^2\bar{s} + 2\frac{d\bar{s}}{dz} + \frac{d^2\bar{s}}{dz^2} + 4n\frac{d\bar{\tau}}{dz} + 4n\bar{\tau} \tag{3.3.27}$$

We first examine the case of $n \geqslant 2$. For this range of values of n , (3.3.26) and (3.3.27) yield the solution

$$\bar{\tau} = A(n)\,e^{nz} + B(n)\,e^{-nz} + C(n)\,e^{(n-2)z} + E(n)\,e^{-(n+2)z} \tag{3.3.28}$$

$$\bar{s} = -2A(n)\,e^{nz} + 2B(n)\,e^{-nz} - 2C(n)\,e^{(n-2)z} + 2E(n)\,e^{-(n+2)z} \tag{3.3.29}$$

Transforming equation (3.3.2) and in view of (3.3.10) we get

$$n\bar{\sigma}_r = n\bar{s} + \frac{d\bar{\tau}}{dz} + 2\bar{\tau} \tag{3.3.30}$$

hence $\bar{\sigma}_r$ is found to be

$$\bar{\sigma}_r = \frac{2-n}{n} A(n) e^{nz} + \frac{2+n}{n} B(n) e^{-nz} - C(n) e^{(n-2)z} + E(n) e^{-(n+2)z} \qquad (3.3.31)$$

Two of the unknown constants can be eliminated from the condition that τ, σ_r and hence $\bar{\tau}$, $\bar{\sigma}_r$ are zero at the inner boundary i.e. at x = 1 or z = 0. Putting z = 0 in (3.3.28) and (3.3.31) we get:

$$A + B + C + E = 0 \qquad (3.3.32a)$$

$$\left(\frac{2}{n} - 1\right) A + \left(\frac{2}{n} + 1\right) B - C + E = 0 \qquad (3.3.32b)$$

Expressing C and E in terms of A and B we get

$$C = -\left(1 - \frac{1}{n}\right) A + \frac{1}{n} B \qquad (3.3.32c)$$

$$E = -\frac{1}{n} A - \left(\frac{1}{n} + 1\right) B \qquad (3.3.32d)$$

Substituting for C and E in (3.3.28) and (3.3.31) we get

$$\bar{\tau} = A(n) \left\{ e^{nz} - \left(1 - \frac{1}{n}\right) e^{(n-2)z} - \frac{1}{n} e^{-(n+2)z} \right\} +$$

$$B(n) \left\{ e^{-nz} + \frac{1}{n} e^{(n-2)z} - \frac{1}{n} e^{-(n+2)z} \right\} \qquad (3.3.33)$$

$$\bar{\sigma}_r = A(n)\left\{\frac{2-n}{n}e^{nz} + \left(1-\frac{1}{n}\right)e^{(n-2)z} - \frac{1}{n}e^{-(n+2)z}\right\} +$$

$$B(n)\left\{\frac{2+n}{n}e^{-nz} - \frac{1}{n}e^{(n-2)z} - \left(1+\frac{1}{n}\right)e^{-(n+2)z}\right\} \quad (3.3.34)$$

Eq. (3.3.33) and (3.3.34) in terms of the independent variable x become

$$\bar{\tau} = A\left\{x^{n} - \left(1-\frac{1}{n}\right)x^{n-2} - \frac{1}{n}x^{-(n+2)}\right\} + B\left\{x^{-n} + \frac{1}{n}x^{n-2} - \left(\frac{1}{n}+1\right)x^{-(n+2)}\right\} \quad (3.3.35)$$

$$\bar{\sigma} = A\left\{\frac{2-n}{n}x^{n} + \left(1-\frac{1}{n}\right)x^{n-2} - \frac{1}{n}x^{-(n+2)}\right\} + B\left\{\frac{2+n}{n}x^{-n} - \frac{1}{n}x^{n-2} - \left(\frac{1}{n}+1\right)x^{-(n+2)}\right\} \quad (3.3.36)$$

also

$$\bar{s} = -2A\left\{x^{n} - \left(1-\frac{1}{n}\right)x^{n-2} + \frac{1}{n}x^{-(n+2)}\right\} + 2B\left\{x^{-n} - \frac{1}{n}x^{n-2} - \left(1+\frac{1}{n}\right)x^{-(n+2)}\right\} \quad (3.3.37)$$

To derive the displacements we note eq. (3.3.12) and (3.3.17) from which:

$$\frac{\partial \bar{u}}{\partial r} = \bar{\epsilon}_r = \frac{\bar{s}}{2G} \quad (3.3.38)$$

and

$$\bar{v} = \frac{1}{n}\left\{r\bar{\epsilon}_\theta - \bar{u}\right\} = -\frac{1}{n}\left\{\frac{r\bar{s}}{2G} + \bar{u}\right\} \tag{3.3.39}$$

Integration of eq. (3.3.38) will introduce a constant, however, substitution in (3.3.12a) for γ and comparison with (3.3.37) shows that the constant must be zero.

Hence on integrating (3.3.38) we find:

$$\bar{u} = -A\frac{r_1}{G}\left\{\frac{x^{n+1}}{n+1} - \frac{x^{n-1}}{n} - \frac{x^{-(n+1)}}{n(n+1)}\right\} - B\frac{r_1}{G}\left\{\frac{x^{1-n}}{n-1} + \frac{1}{n(n-1)}x^{n-1} - \frac{1}{n}x^{-(n+1)}\right\} \tag{3.3.40}$$

Also (3.3.39) yields,

$$\bar{v} = \frac{Ar_1}{G}\left\{\frac{n+2}{n(n+1)}x^{n+1} - \frac{1}{n}x^{n-1} + \frac{1}{n(n+1)}x^{-(n+1)}\right\} + B\frac{r_1}{G}\left\{-\frac{n-2}{n(n-1)}x^{1-n} + \frac{1}{n(n-1)}x^{n-1} + \frac{1}{n}x^{-(n+1)}\right\} \tag{3.3.41}$$

For convenience we write (3.3.35) in the form:

$$\bar{\tau} = A(n)\psi_{1n}(x) + B_n\psi_{2n}(x) \tag{3.3.42}$$

and (3.3.36) as

$$\bar{\sigma}_r = A(n)\,\phi_{1n}(x) + B(n)\,\phi_{2n}(x) \tag{3.3.43}$$

where

$$\psi_{1n}(x) = x^{n} - \left(1-\frac{1}{n}\right)x^{n-2} - \frac{1}{n}\,x^{-(n+2)} \tag{3.3.44a}$$

$$\psi_{2n}(x) = x^{-n} + \frac{1}{n}\,x^{n-2} - \left(1+\frac{1}{n}\right)x^{-(n+2)} \tag{3.3.44b}$$

$$\phi_{1n}(x) = \frac{2-n}{n}\,x^{n} + \left(1-\frac{1}{n}\right)x^{n-2} - \frac{1}{n}\,x^{-(n+2)} \tag{3.3.44c}$$

$$\phi_{2n}(x) = \frac{2+n}{n}\,x^{-n} - \frac{1}{n}\,x^{n-2} - \left(1+\frac{1}{n}\right)x^{-(n+2)} \tag{3.3.44d}$$

Also we write (3.3.40) as

$$\bar{u} = -A(n)\,\frac{r_1}{G}\,U_{1n}(x) - B(n)\,\frac{r_1}{G}\,U_{2n}(x) \tag{3.3.45}$$

and (3.3.41) as

$$\bar{v} = A(n)\,\frac{r_1}{G}\,V_{1n}(x) + B(n)\,\frac{r_1}{G}\,V_{2n}(x). \tag{3.3.46}$$

where

$$U_{1n}(x) = \frac{1}{n+1}\,x^{n+1} - \frac{1}{n}\,x^{n-1} - \frac{1}{n(n+1)}\,x^{-(n+1)} \tag{3.3.47a}$$

$$U_{2n}(x) = \frac{1}{n-1}x^{1-n} + \frac{1}{n(n-1)}x^{n-1} - \frac{1}{n}x^{-(n+1)} \tag{3.3.47b}$$

$$V_{1n}(x) = \frac{n+2}{n(n+1)}x^{n+1} - \frac{1}{n}x^{n-1} + \frac{1}{n(n+1)}x^{-(n+1)} \tag{3.3.47c}$$

$$V_{2n}(x) = -\frac{n-2}{n(n-1)}x^{1-n} + \frac{1}{n(n-1)}x^{n-1} + \frac{1}{n}x^{-(n+1)} \tag{3.3.47d}$$

The constants $A(n)$ and $B(n)$ cannot be evaluated since neigher the stresses nor the displacements are known at the cylinder-shell interface.

For the case of $n = 1$ equs. (3.3.26) and (3.3.27) become:

$$\bar{s}_1 + \frac{d\bar{s}_1}{dz} + \frac{d^2\bar{\tau}_1}{dz^2} + 2\frac{d\bar{\tau}_1}{dz} + \bar{\tau}_1 = 0 \tag{3.3.48}$$

$$\frac{d^2\bar{s}_1}{dz^2} + 2\frac{d\bar{s}_1}{dz} + \bar{s}_1 + 4\frac{d\bar{\tau}_1}{dz} + 4\bar{\tau}_1 = 0 \tag{3.3.49}$$

where

$$\bar{s}_1 = \bar{s}(z,1) \qquad \bar{\tau}_1 = \bar{\tau}(z,1) \tag{3.3.50}$$

Eq. (3.3.48) and (3.3.49) yield the solution in the x variable

$$\bar{\tau}_1 = A_1 x + B_1 x^{-3} \tag{3.3.51}$$

$$\bar{S}_1 = -2A_1 x + 2B_1 \bar{x}^3 + E_1 \bar{x}^1 \qquad (3.3.52)$$

Transforming (3.3.2) for $n = 1$ and using (3.3.10) we get

$$\bar{\sigma}_{r_1} = \bar{S}_1 + \bar{\tau}_1 + \frac{d}{dx}(x \bar{\tau}_1) - \rho g r_1 x \qquad (3.3.53)$$

and substituting for $\bar{S}_1$ and $\bar{\tau}_1$ we get

$$\bar{\sigma}_{r_1} = A_1 x + B_1 \bar{x}^3 + E_1 \bar{x}^1 - \rho g r_1 x \qquad (3.3.54)$$

Using (3.3.38) and (3.3.39) for $n = 1$ we get

$$\bar{u}_1 = \frac{r_1}{2G} \left\{ -A_1 x^2 - B_1 \bar{x}^2 + E_1 \log x + D_1 \right\} \qquad (3.3.55)$$

$$\bar{v}_1 = \frac{r_1}{2G} \left\{ 3A_1 x^2 - B_1 \bar{x}^2 - E_1 (\log x + 1) - D_1 \right\} \qquad (3.3.56)$$

at $x = 1$, $\bar{\tau}_1 = 0$, $\bar{\sigma}_{r_1} = 0$ (3.3.57)

hence:

$$A_1 + B_1 = 0 \qquad (3.3.58)$$

$$A_1 + B_1 + E_1 - \rho g r_1 = 0 \qquad (3.3.59)$$

$$\therefore \quad A_1 = -B_1 \quad \text{and} \quad E = \rho g r_1 \qquad (3.3.60)$$

and

$$\bar{\tau}_1 = A_1(x - x^{-3}) \tag{3.3.61}$$

$$\bar{\sigma}_{r_1} = A_1(x - x^{-3}) - \rho g r_1 \left(x - \frac{1}{x}\right) \tag{3.3.62}$$

$$\bar{u}_1 = \frac{r_1}{2G}\left\{-A_1 x^2 + A_1 x^{-2} + \rho g \log x + D_1\right\} \tag{3.3.63}$$

$$\bar{v}_1 = \frac{r_1}{2G}\left\{A_1(3x^2 + x^{-2}) - \rho g(1 + \log x) - D_1\right\} \tag{3.3.64}$$

We finally deal with the case of n = 0. We make use of (3.3.13) i.e.

$$\epsilon_r + \epsilon_\theta = 0 \quad , \quad \frac{d\bar{u}_0}{dr} + \frac{\bar{u}_0}{r} = 0 \tag{3.3.65}$$

$$\bar{u}_0 = \frac{A_0'}{r} \tag{3.3.66}$$

$$\bar{\epsilon}_{r0} = -\frac{A_0'}{r^2} \quad , \quad \bar{\epsilon}_{\theta_0} = \frac{A_0'}{r^2} \tag{3.3.67}$$

$$\bar{S}_0 = -2G\frac{A_0'}{r^2} \tag{3.3.68}$$

From (3.3.1)

$$\frac{d\bar{\sigma}_{r_0}}{dr} + \frac{\bar{S}_0}{r} = 0 \tag{3.3.69}$$

hence:

$$\frac{d\bar{\sigma}_{r0}}{dr} = \frac{2G}{r^3} A_0' \tag{3.3.70}$$

$$\bar{\sigma}_{r0} = A_0'' - \frac{G A_0'}{r^2} \tag{3.3.71}$$

at $r = r_1$

$$\bar{\sigma}_{r0} = 0 \tag{3.3.72}$$

Let

$$\frac{G A_0'}{r_1^2} = A_0 \tag{3.3.73}$$

Then

$$\bar{\sigma}_{r0} = A_0 \left(1 - \frac{1}{x^2}\right) \tag{3.3.74}$$

$$\bar{u}_0 = A_0 \frac{r_1}{G} x^{-1} \tag{3.3.75}$$

Once $A(n)$ and $B(n)$ have been found from the stress and displacement continuity conditions at the cylinder-shell interface then

$$\tau = \sum_{n=1}^{\infty} \bar{\tau}(x, n) \sin n\theta \tag{3.3.76}$$

$$\sigma_r = \sigma_{r0} + \sum_{n=1}^{\infty} \bar{\sigma}_r(x, n) \cos n\theta \tag{3.3.77}$$

$$u = u_0 + \sum_{n=1}^{\infty} \bar{u}(x,n) \cos n\theta \tag{3.3.78}$$

$$v = \sum_{n=1}^{\infty} \bar{v}(x,n) \sin n\theta \tag{3.3.79}$$

3.4 Analysis of the Containing Elastic Shell

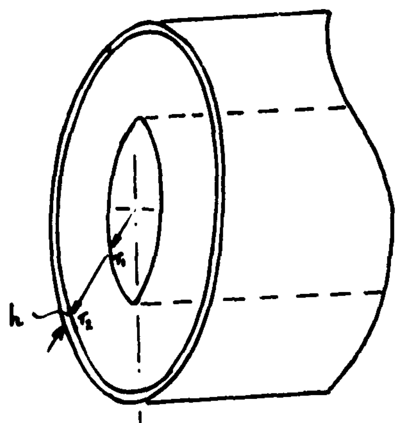
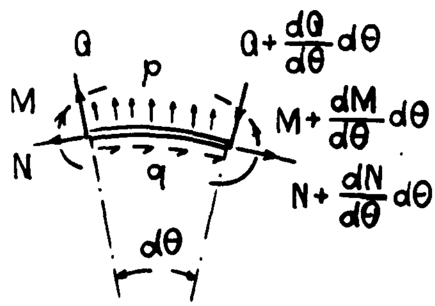


Figure (3.4.1)



The equilibrium relations for a shell element are:

In the tangential direction

$$Q_\theta = \frac{dN_\theta}{d\theta} + r_2 q \tag{3.4.1}$$

In the normal direction

$$\frac{dQ_\theta}{d\theta} + N_\theta = r_2 p \tag{3.4.2}$$

and the moment equilibrium is

$$Q_\theta = \frac{1}{r_2}\frac{dM_\theta}{d\theta} \tag{3.4.3}$$

Eliminating Q_θ from (3.4.1), (3.4.2) and (3.4.3) we get dropping the suffix θ

$$\frac{d^2N}{d\theta^2} + r_2\frac{dq}{d\theta} + N = r_2 p \tag{3.4.4}$$

$$\frac{dN}{d\theta} + r_2 q = \frac{1}{r_2}\frac{dM}{d\theta} \tag{3.4.5}$$

$$\frac{1}{r_2}\frac{d^2M}{d\theta^2} + N = r_2 p \tag{3.4.6}$$

The force displacement relati on becomes

$$N = \frac{E_s h}{1-\nu_s^2}\left\{\frac{1}{r_2}\frac{dv_s}{d\theta} + \frac{u_s}{r_2}\right\} \tag{3.4.7}$$

$$M = \frac{D}{r_2^2}\left\{\frac{d^2u_s}{d\theta^2} - \frac{dv_s}{d\theta}\right\} \tag{3.4.8}$$

where

$$D = \frac{E_s h^3}{12(1-\nu_s^2)} \tag{3.4.9}$$

Taking Fourier transform of (3.4.4,5a, 6) we get

$$-n^2\bar{N} + r_2 n\bar{q} + \bar{N} = r_2\bar{p} \qquad (3.4.4a)$$

$$-n\bar{N} + r_2\bar{q} = -\frac{n}{r_2}\bar{M} \qquad (3.4.5a)$$

$$-\frac{n^2}{r_2}\bar{M} + \bar{N} = r_2\bar{p} \qquad (3.4.6a)$$

Also taking Fourier's transform of (3.4.7) and (3.4.8) we obtain

$$\bar{N} = C^*\left\{n\bar{v}_s + \bar{u}_s\right\} \qquad (3.4.7a)$$

$$\bar{M} = r_2 D^*\left\{-n^2\bar{u}_s - n\bar{v}_s\right\} \qquad (3.4.8a)$$

where $D^* = \dfrac{D}{r_2^{\,3}}$, $C^* = \dfrac{E_s}{1-\nu_s^2}\left(\dfrac{h}{r_2}\right)$

Substituting for $\bar{M}$ and $\bar{N}$ in (3.4.5a) we get

$$r_2\bar{q} = nC^*\left\{n\bar{v}_s + \bar{u}_s\right\} + nD^*\left\{n^2\bar{u}_s + n\bar{v}_s\right\}$$

or

$$r_2\bar{q} = \bar{u}_s\left\{nC^* + n^3D^*\right\} + \bar{v}_s\left\{n^2C^* + n^2D^*\right\} \qquad (3.4.10)$$

Similarly substituting for $\bar{M}$ and $\bar{N}$ in (3.4.6a)

$$r_2 \bar{p} = n^2 D^* \left\{ n^2 \bar{u}_s + n \bar{v}_s \right\} + C^* \left\{ n \bar{v}_s + \bar{u}_s \right\}$$

or,

$$r_2 \bar{p} = \left\{ n^4 D^* + C^* \right\} \bar{u}_s + \left\{ n^3 D^* + n C^* \right\} \bar{v}_s \qquad (3.4.11)$$

For continuity of stress at the cylinder-shell interface we must have (Fig. 3.4.2):

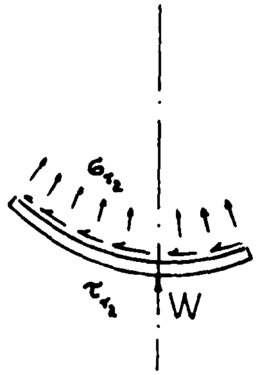


Figure (3.4.2)

$$p(\theta) = - \sigma_r(r_2,\theta) - W(\theta) + w_s \cos\theta \qquad (3.4.12)$$

$$q(\theta) = - w_s \sin\theta - \tau(r_2, \theta) \qquad (3.4.13)$$

where w_s is the weight of unit length of shell per unit length of arc. Hence:

$$\bar{p} = \begin{cases} -\bar{\sigma}_r(r_2) - \dfrac{W}{\pi r_2} , & n \neq 1 \\ -\bar{\sigma}_r(r_2) - \dfrac{W}{\pi r_2} + \omega_s , & n = 1 \end{cases} \tag{3.4.14}$$

$$\bar{q} = \begin{cases} -\bar{\tau}(r_2) , & n \neq 1 \\ -\bar{\tau}(r_2) - \omega_s , & n = 1 \end{cases} \tag{3.4.15}$$

Also continuity of displacements at the interface requires that

$$u(r_2) = u_s \qquad \bar{u}(r_2) = \bar{u}_s \tag{3.4.16}$$

$$v(r_2) = v_s \qquad \bar{v}(r_2) = \bar{v}_s \tag{3.4.17}$$

Case of $n \geq 2$. Let $\left(\frac{r_2}{r_1}\right) = a$.

From (3.4.15), (3.3.42) and (3.4.10) it can be seen that

$$-\left\{A\psi_{1n}(a) + B\psi_{2n}(a)\right\} r_2 = \bar{u}_s\left\{nC^* + n^3 D^*\right\} + \bar{v}_s n^2\left(C^* + D^*\right) \tag{3.4.18}$$

$$-\frac{W}{\pi} - r_2\left\{A\phi_{1n}(a) + B\phi_{2n}(a)\right\} = \bar{u}_s\left\{C^* + n^4 D^*\right\} + \bar{v}_s\left\{nC^* + n^3 D^*\right\} \tag{3.4.19}$$

Also from (3.3.45) and (3.3.46)

$$\bar{u}_s = -A\frac{r_1}{G}U_{1n}(a) - B\frac{r_1}{G}U_{2n}(a) \qquad (3.4.20)$$

$$\bar{v}_s = A\frac{r_1}{G}V_{1n}(a) + B\frac{r_1}{G}V_{2n}(a) \qquad (3.4.21)$$

Solving for A and B from (3.4.20) and (3.4.21).

$$A(n) = \frac{G}{r_1}\left\{\bar{u}_s V_{2n}' + \bar{v}_s U_{2n}'\right\} \qquad (3.4.22)$$

$$B(n) = \frac{G}{r_1}\left\{\bar{u}_s V_{1n}' + \bar{v}_s U_{1n}'\right\} \qquad (3.4.23)$$

where

$$V_{1n}' = V_{1n}(a)\left\{U_{2n}(a)V_{1n}(a) - U_{1n}(a)V_{2n}(a)\right\}^{-1} \qquad (3.4.24)$$

$$U_{1n}' = U_{1n}(a)\left\{U_{2n}(a)V_{1n}(a) - U_{1n}(a)V_{2n}(a)\right\}^{-1} \qquad (3.4.25)$$

etc.

Substituting for A and B in (3.4.18) and (3.4.19) and collecting terms we get:

$$\bar{u}_s\left\{G\left[\psi_{2n}(a)V_{1n}'(a) - \psi_{1n}(a)V_{2n}'(a)\right] - n^3D^* - nC^*\right\} +$$

$$\bar{v}_s\left\{G\left[\psi_{2n}(a)U_{1n}'(a) - \psi_{1n}(a)U_{2n}'\right] - n^2(D^* + C^*)\right\} = 0 \qquad (3.4.26)$$

$$\bar{u}_s \left\{ G\left[\phi_{2n}(a) V_{1n}'(a) - \phi_{1n}(a) V_{2n}'(a) \right] - n^4 D^* - C^* \right\} +$$

$$+ \bar{v}_s \left\{ G\left[\phi_{2n}(a) U_{1n}'(a) - \phi_{1n}(a) U_{2n}'(a) \right] - n^3 D^* - n C^* \right\} = \frac{W}{\pi} \quad (3.4.27)$$

From equations (3.4.26) and (3.4.27) $\bar{u}_s$ and $\bar{v}_s$ may be found.

Substitution of (3.4.26) and (3.4.27) in (3.4.22) and (3.4.23) gives $A(n)$ and $B(n)$ respectively.

<u>Case of n = 1</u>

Equations (3.4.10) and (3.4.11) become:

$$r_2 \bar{q}_1 = C^* \left\{ \bar{v}_{s_1} + \bar{u}_{s_1} \right\} + D^* \left\{ \bar{v}_{s_1} + \bar{u}_{s_1} \right\} \quad (3.4.28)$$

$$r_2 \bar{p}_1 = C^* \left\{ \bar{v}_{s_1} + \bar{u}_{s_1} \right\} + D^* \left\{ \bar{v}_{s_1} + \bar{u}_{s_1} \right\} \quad (3.4.29)$$

hence:

$$\bar{q}_1 = \bar{p}_1 = \frac{1}{r_2} \left\{ C^* + D^* \right\} \left\{ \bar{u}_{s_1} + \bar{v}_{s_1} \right\} \quad (3.4.30)$$

The relation $\bar{q}_1 = \bar{p}_1$ satisfies the condition of vertical equilibrium, i.e.

$$\int_{-\pi}^{\pi} q \sin\theta \, d\theta = \int_{-\pi}^{\pi} p \cos\theta \, d\theta \quad (3.4.31)$$

also for n = 1 from (3.3.6) and (3.3.62)

$$\bar{\tau} = A_1 \{x - x^{-3}\} \tag{3.3.61}$$

$$\bar{\sigma}_r = A_1 \{x - x^{-3}\} - \rho g r_1 (x - \tfrac{1}{x}) \tag{3.3.62}$$

In view of these and also (3.4.14) and (3.4.15)

$$\bar{q}_1 = - A_1 (a - a^{-3}) - w_s \tag{3.4.32}$$

$$\bar{p}_1 = - A_1 (a - a^{-3}) + \rho g r_1 (a - \tfrac{1}{a}) + w_s - \frac{W}{\pi r_2} \tag{3.4.33}$$

Equations (3.3.63) and (3.3.64) can only be consistent in view of $\bar{p}_1 = \bar{q}_1$ if

$$W = 2\pi r_2 w_s + \rho g \pi r_1 r_2 \left(\frac{r_2}{r_1} - \frac{r_1}{r_2}\right) \tag{3.4.34}$$

which is true, since the right hand side represents the combined weight of shell and cylinder.

Also from (3.3.63) and (3.3.64) we have

$$\bar{u}_{s_1} = \frac{r_1}{2G} \left\{ - A_1 (a^2 - a^{-2}) + \rho g \log a + D_1 \right\} \tag{3.4.35}$$

$$\bar{v}_{s_1} = \frac{r_1}{2G}\left\{3A_1 a^2 + A_1 \bar{a}^2 - pg \log a - pg - D_1'\right\} \qquad (3.4.36)$$

Adding these last two equations we get:

$$\bar{u}_{s_1} + \bar{v}_{s_1} = \frac{r_1}{G}\left\{A_1 a^2 + A_1 \bar{a}^2 - \frac{pg}{2}\right\} \qquad (3.4.37)$$

Hence:

$$A_1 = \left\{a^2 + \bar{a}^2\right\}^{-1}\left\{\frac{G}{r_1}(\bar{u}_{s_1} + \bar{v}_{s_1}) + \frac{pg}{2}\right\} \qquad (3.4.38)$$

also from(3.4.32)

$$A_1 = -\left\{\frac{1}{r_2}\{C^* + D^*\}(\bar{u}_{s_1} + \bar{v}_{s_1}) + \omega_s\right\}(a - \bar{a}^3)^{-1} \qquad (3.4.39)$$

From (3.4.38) and (3.4.39),

$$\frac{1}{r_1}(\bar{u}_{s_1} + \bar{v}_{s_1})\left\{\frac{G}{a^2 + \bar{a}^2} + \frac{C^* + D^*}{a^2 - a^{-2}}\right\} = -\frac{1}{2}pg(a^2 + \bar{a}^2)^{-1} - \frac{a\omega_s}{a^2 - \bar{a}^2} \qquad (3.4.40)$$

Otherwise by eliminating ($\bar{u}_{s_1} + \bar{v}_{s_1}$) we can determine A_1 i.e.

or

$$-A_1(a-a^{-3}) = \omega_s + \frac{1}{a}\,\frac{C^*+D^*}{G}\left\{A_1(a^2+a^{-2})-\frac{1}{2}\rho g\right\} \qquad (3.4.41)$$

Naturally, for the moment, $\bar{u}_{s_1} - v_{s_1}$ cannot be determined since it represents a rigid body displacement. Finally, however, $\bar{u}_{s_1}$ will be determined from the condition that $u_s = 0$ at $\Theta = 0$.

<u>Case of n = 0</u>

From (3.4.11)

$$r_2\,\bar{p}_0 = \bar{u}_{s_0}\,C^* \qquad (3.4.42)$$

also in view of (3.4.14)

$$\bar{p}_0 = -\sigma_{r0} - \frac{W}{\pi r_2} \qquad (3.4.43)$$

$$\bar{p}_0 = -A_0\left(1-\frac{1}{a^2}\right) - \frac{W}{\pi r_2} \qquad (3.4.43a)$$

Hence:

$$\bar{u}_{s_0}\,C^* = -A_0 r_2\left(1-\frac{1}{a^2}\right) - \frac{W}{\pi} \qquad (3.4.44)$$

But from (3.3.76)

$$\bar{u}_{s_0} = A_0\,\frac{r_1}{G}\cdot\frac{1}{a} \qquad (3.3.75)$$

or

$$A_o = \frac{a}{r_1} G \bar{u}_{so} \tag{3.4.45}$$

Substituting for A_o in (3.4.44) we get

$$\bar{u}_{so}\left\{ C^* + a^2 G\left(1 - \frac{1}{a^2}\right)\right\} = -\frac{W}{\pi}$$ or

$$\bar{u}_{so} = -\frac{W}{\pi}\left\{C^* + G(a^2-1)\right\}^{-1} \tag{3.4.46}$$

also from (3.4.45)

$$A_o = -\frac{aWG}{\pi r_1}\left\{C^* + G(a^2-1)\right\}^{-1} \tag{3.4.47}$$

This completes the formal solution to the elastic problem.

3.5 The Associate Viscoelastic Problem

To obtain the solution to the viscoelastic problem we make use of the integral form of the constitutive relation, i.e.

$$s_{ij} = \int_{-\infty}^{t} G(t-\tau)\frac{\partial e_{ij}}{\partial \tau}\, d\tau \tag{3.5.1}$$

Upon taking Laplace transform with respect to time (2.3.1) becomes:

$$\underline{s}_{ij} = p\underline{G}\, \underline{e}_{ij} \tag{3.5.2}$$

where the dash designates a transformed quantity. In view of (3.5.2) it is evident that the solution to the viscoelastic problem can be obtained on substitution of $\underline{G}\, p$ for G in the relevant equations and taking the inverse transform [1].

The numerical analysis of the problem consists in calculating the displacements of the cylinder both at the inner and outer radius. These results will be presented at a later date.

APPENDIX I

Numerical Applications

Infinite Viscoelastic Slab

To test their accuracy, the methods outlined in Chapter I were applied to the problem of a viscoelastic slab, which has been solved exactly, within the accuracy of the numerical computations employed, by Muki and Sternberg .

The slab is of infinite extent and finite thickness 2a, bounded by the planes $z = \pm a$. Initially it is at uniform temperature, 80°C, when suddenly the temperature of the faces changes to 110°C and remains constant at that value.

The solution of the problem reduces to

$$\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \ , \ \sigma_z = 0 \ , \ \sigma_x = \sigma_y = f(z, t) \tag{A.1.1}$$

$$\epsilon_{xy} = \epsilon_{yz} = \epsilon_{zx} = 0 \ , \ \epsilon_z = \epsilon_z(z, t) \ , \ \epsilon_x = \epsilon_y = 0 \tag{A.1.2}$$

where σ_x is found from the integral equation

$$\sigma_x + \frac{1}{3K}\int_0^t E(\xi - \xi')\frac{\partial \sigma_x}{\partial \tau}\, d\tau = -2\alpha_0 \int_0^t E(\xi - \xi')\frac{\partial \Theta}{\partial \tau}\, d\tau \tag{A.1.3}$$

In Ref. [7] the problem was solved by referring (A.1.3) to the plane and taking Laplace transform. However, in view of the

realistic nature of $E(t)$, finding the analytic representation of the latter, taking Laplace transform of (A.1.3) and inverting was a very tedious procedure.

Here we obtain a solution at the middle plane of the slab first by reducing (A.1.3) to a set of linear algebraic equations by the method described in Chapter I.

Let σ_x be piecewise linear in t i.e. in the interval

$$t_{K-1} \leq t \leq t_K$$

$$\sigma_x = \sigma_{x_{K-1}} + A_K (t - t_{K-1}) \tag{A.1.4}$$

Also let

$$A_K (t_K - t_{K-1}) = a_K \tag{A.1.5}$$

Then in matrix form (A.1.3) becomes

$$[H]\{a\} + \frac{1}{3K}[E]\{a\} = \{F\} \tag{A.1.6}$$

where F is a known function.

From (A.1.6)

$$\{a\} = [R]^{-1}\{F\} \tag{A.1.7}$$

where $$[R] = [H] + \frac{1}{3K}[E] \tag{A.1.8}$$

and $$\{\sigma_x\} = [H]\{a\} \tag{A.1.9}$$

Referring to the data of Ref.   for the material properties we can find:

$$E = \begin{bmatrix} 2.24 \\ 2.01 & 2.24 \\ 1.79 & 1.88 & 2.24 \\ 1.52 & 1.61 & 1.69 & 2.24 \\ 1.29 & 1.33 & 1.39 & 1.44 & 2.24 \\ 1.00 & 1.04 & 1.08 & 1.11 & 1.15 & 2.24 \\ .80 & .80 & .80 & .80 & .80 & .80 & 2.24 \end{bmatrix} .10^{10} \qquad (A.1.9a)$$

$$R = \begin{bmatrix} 1.2933 \\ 1.2632 & 1.2933 \\ 1.2344 & 1.2462 & 1.2933 \\ 1.1991 & 1.2019 & 1.2213 & 1.2933 \\ 1.1689 & 1.1742 & 1.1820 & 1.1886 & 1.2933 \\ 1.1397 & 1.1362 & 1.1414 & 1.1454 & 1.1506 & 1.2933 \\ 1.1048 & 1.1048 & 1.1048 & 1.1048 & 1.1048 & 1.1048 & 1.2933 \end{bmatrix} .10^{10} \qquad (A.1.9b)$$

(A.1.9) being the solution to the problem. The results obtained by this method have been plotted for comparison against the "exact" solution obtained in Ref. [7] - See Fig. (A.1.4)- and they compare very favorably, in view of the fact that only a desk calculator was used for the numerical computations.

The iterative method was also used to solve (A.1.3) again at the middle plane $z = 0$. The zero'th approximation to the solution was found by putting $K = \infty$. Then

$$\sigma_{x_0} = -2\alpha_0 \int_0^t E(\xi - \xi') \frac{\partial \Theta}{\partial \tau} d\tau \tag{A.1.10}$$

Iteration was then carried out on the basis of the formula

$$\sigma_{x_n} + \frac{1}{3K} \int_0^t E(\xi - \xi') \frac{\partial \sigma_{x,n-1}}{\partial \tau} d\tau =$$

$$= -2\alpha_0 \int_0^t E(\xi - \xi') \frac{\partial \Theta}{\partial \tau} d\tau \tag{A.1.11}$$

were in iterations subsequent to the zero'th the actual value of was used. In Fig. (A.1.4), it is shown that after two iterations the solution obtained was practically - within error of numerical computations involved - identical with the "exact solution" cited above.

The above iteration procedure is in fact convergent as can be seen by the following simple proof.

Write $$E(\xi - \xi') = E_0\, \mathcal{E}(\xi - \xi') \tag{A.1.12}$$

where $$\mathcal{E}(\xi - \xi') \leq 1 \tag{A.1.13}$$

$$\frac{E_o}{3K} = 1-2\nu_o = \lambda < 1 \quad \text{for } \nu_o > 0 \qquad \text{(A.1.14)}$$

Equation (A.1.8) becomes:

$$\sigma_x + \lambda \int_0^t \mathcal{E}(\xi-\xi') \frac{\partial \sigma_x}{\partial \tau} d\tau = -F(t)$$

where

$$F(t) = \int_0^t 2\alpha_o \, E(\xi-\xi') \frac{\partial \Theta}{\partial \tau} d\tau \qquad \text{(A.1.15)}$$

Furthermore $F(t)$ is bounded in every interval, $0 \leq \tau \leq t$, See Fig. (A.1.4). Hence

$$|F(t)| \leq M \qquad \text{(A.1.16)}$$

Then

$$|\sigma_{x_0}| \leq M \qquad \text{(A.1.17)}$$

$$\sigma_{x_1} = -F(t) - \int_0^t \mathcal{E}(\xi-\xi') \frac{\partial F}{\partial \tau} d\tau \qquad \text{(A.1.18)}$$

and

$$|\sigma_{x_1}| \leq |F(t)|_{max} + \lambda |\mathcal{E}|_{max} |F(t)|_{max} \qquad \text{(A.1.19)}$$

or

$$|\sigma_{x_1}| \leq M + \lambda M \qquad \text{(A.1.20)}$$

also

$$|\sigma_{x_2}| \leq M + \lambda M + \lambda^2 M \qquad \text{(A.1.21)}$$

and

$$\sigma_{x_n} \leq M\left(1+\lambda+\lambda^2+\cdots+\lambda^n\right) \qquad \text{(A.1.22)}$$

The series $1+\lambda+\cdots+\lambda^n$ is convergent for $\lambda < 1$ and equals $\frac{1}{1-\lambda}$

Hence

$$|\sigma x_n| < \frac{M}{1-\lambda} \qquad \text{(A.1.23)}$$

and the iteration is convergent. Also

$$\left| \sigma_{x_n}| - |\sigma_{x_{n-1}} \right| \leq M\lambda^n \underset{n \to \infty}{\longrightarrow} 0 \qquad \text{(A.1.24)}$$

Hence solution can be approached as closely as we please, by a sufficient number of iterations.

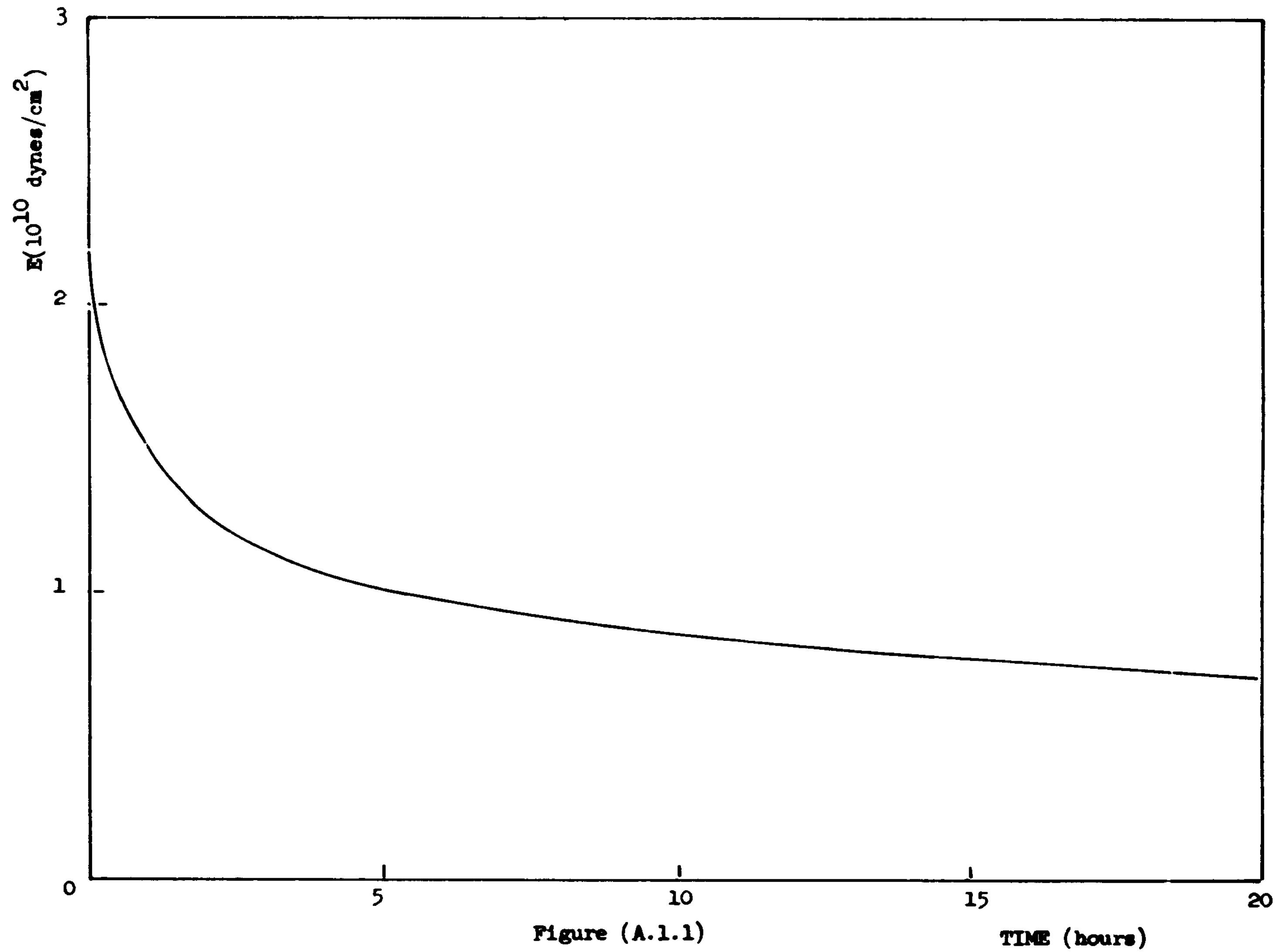


Figure (A.1.1)

TENSION MODULUS E FOR METHYL METHACRYLATE

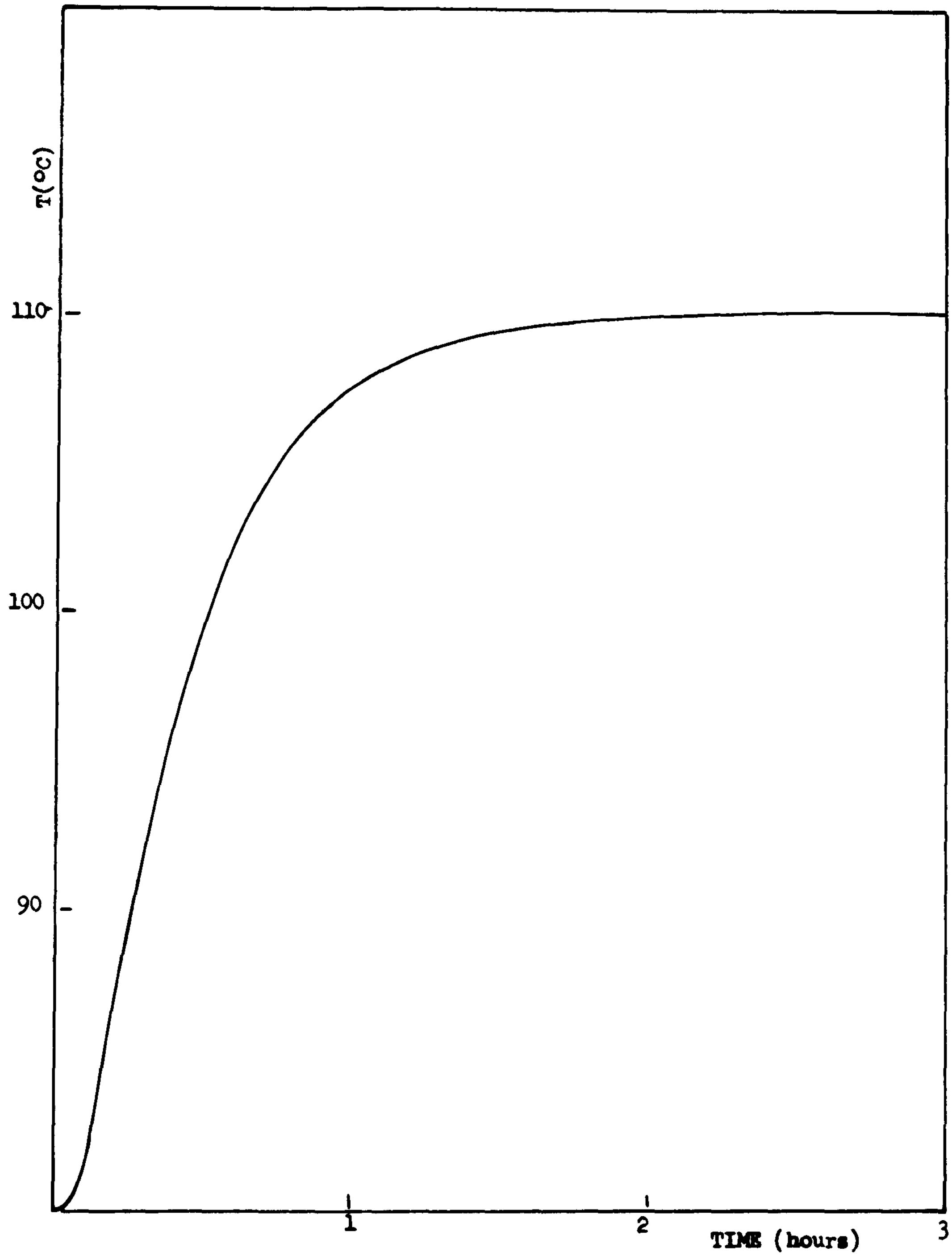


Figure (A.1.2)

TEMPERATURE AT MIDDLE PLANE OF SLAB

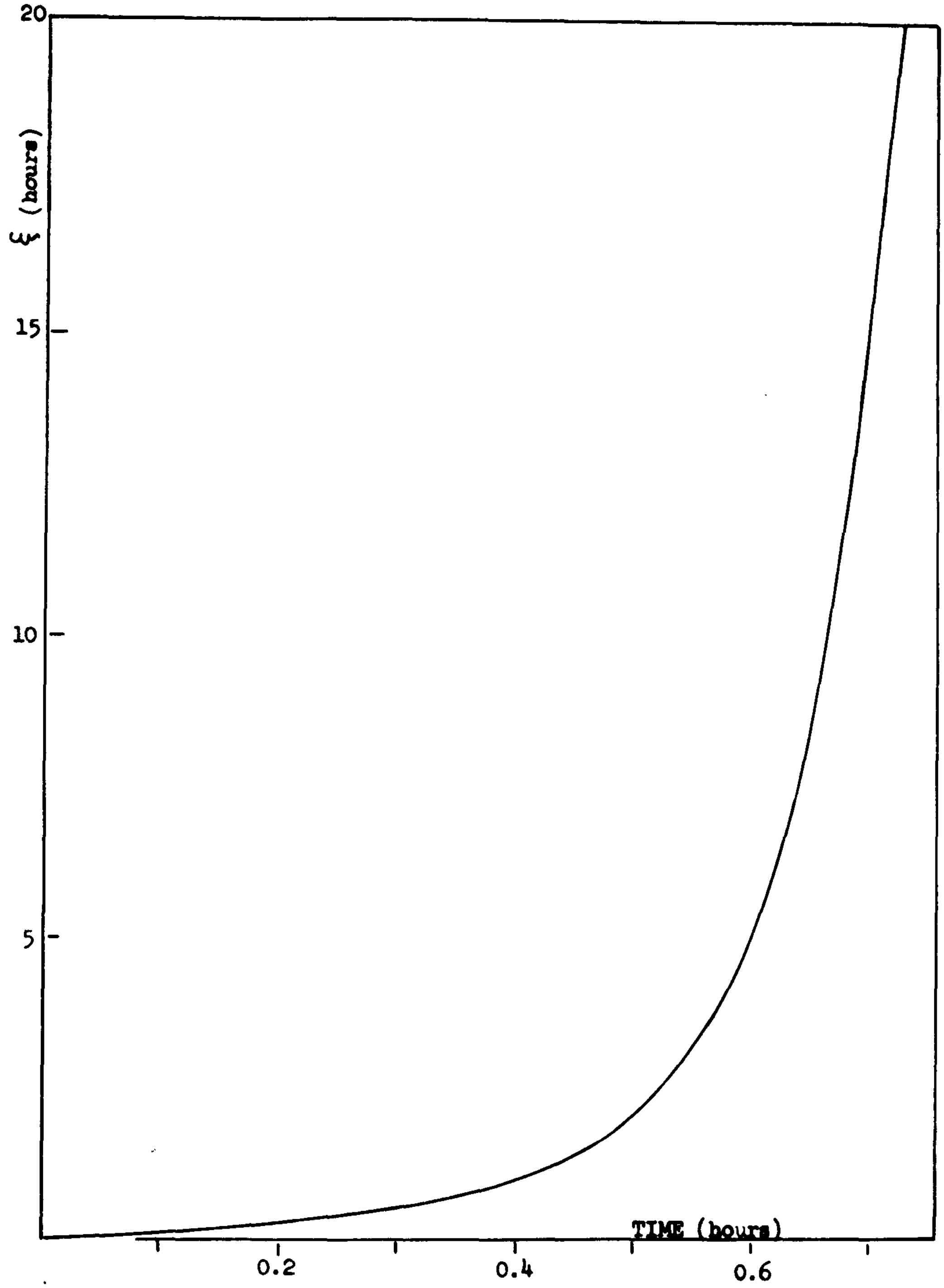


Figure (A.1.3)

REDUCED TIME COORDINATE

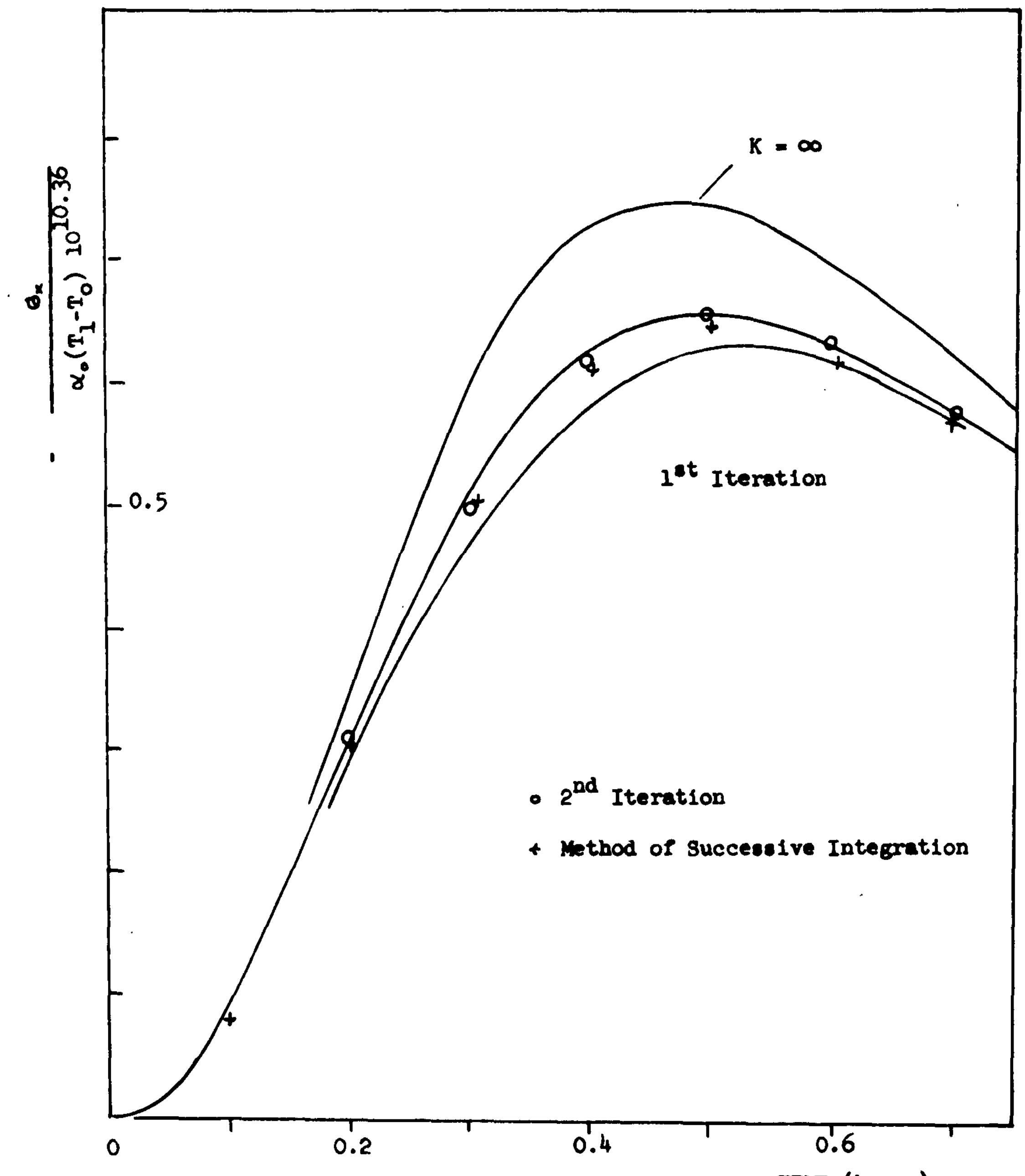


Figure (A.1.4)

STRESS IN MIDDLE PLANE OF THE SLAB

Viscoelastic Sphere With a Step Rise in Surface Temperature

The same methods, under more unfavorable circumstances of non-zero initial conditions and rapid changes in temperature, are now used to obtain the hoop stress at the surface of a solid viscoelastic sphere, subjected to a step temperature rise at the surface. It is found that in applying the first method some care must be taken near t = o. Otherwise both methods give results which are in excellent agreement with the exact solution of Ref.[7] .

Muki and Sternberg [7] obtained an exact solution for the stresses by assuming that the material has an elastic bulk modulus.

Using their notation, S_r can be obtained by solving the following integral equation,

$$S_r(r,t) + \frac{2}{3K}\int_0^t G(\xi-\xi')\frac{\partial S_r}{\partial \tau}\,d\tau = \int_0^t G(\xi-\xi')\frac{\partial h}{\partial \tau}\,d\tau \qquad \text{(A.1.25)}$$

whilst on the surface of the sphere

$$\sigma_\theta(r_2,t) = -\frac{3}{2}\,S_r(r_2,t) \qquad \text{(A.1.26)}$$

When (A.1.25) is referred to the (r, ξ) plane it becomes

$$\hat{S}_r(r,\xi) + \frac{2}{3K}\int_0^\xi G(\xi-\xi')\frac{\partial \hat{S}_r}{\partial \xi'}\,d\xi' = \int_0^\xi G(\xi-\xi')\frac{\partial \hat{h}}{\partial \xi'}\,d\xi' \qquad \text{(A.1.27)}$$

where

$$\hat{S}_r(\xi,r) = S_r\{t(\xi,r),r\}\ ,\quad \hat{h}(\xi,r) = h\{t(\xi,r),r\} \qquad \text{(A.1.28)}$$

Now taking Laplace transform of (A.1.28) with respect to ξ , , one obtains:

$$\bar{\hat{s}}_{r} \left\{ 1 + \frac{2\bar{G}p}{3K} \right\} = p\bar{G}\,\bar{\hat{h}} \qquad (A.1.29)$$

or

$$\bar{\hat{s}}_{r} = \frac{3K\bar{G}}{3K+2\bar{G}p}\, p\bar{\hat{h}} = \bar{R}\, p\,\bar{\hat{h}} \qquad (A.1.30)$$

where R is then obtained by Laplace inversion.

It is shown however in the General Introduction

$$\bar{G} = \frac{6K\bar{E}}{9K-p\bar{E}} \qquad (A.1.31)$$

where E(t) the tension modulus of the material.

On substitution of (A.1.31) in (A.1.30) we obtain:

$$\bar{\hat{s}}_{r}(r,p) = \frac{2K\bar{E}}{3K+p\bar{E}}\, p\,\bar{\hat{h}}(r,p) \qquad (A.1.32)$$

On taking inverse Laplace transform of (A.1.32) and reverting to the (r,t) plane (A.1.32) becomes:

$$s_{r}(r,t) + \frac{1}{3K}\int_0^t E(\xi-\xi')\frac{\partial s_r}{\partial \tau}\,d\tau =$$

$$= \int_0^t E(\xi-\xi')\frac{\partial h}{\partial \tau}\,d\tau \qquad (A.1.33)$$

On the other hand from (A.1.30):

$$S_{r}(r,t) = \int_{0}^{t} R(\xi-\xi')\,\frac{\partial h}{\partial \tau}\,d\tau \qquad \text{(A.1.34)}$$

Naturally, since R is known a solution for S_{r} can be obtained from (A.1.34) by simple quadratures. Such a solution provides a basis for comparison with the two suggested methods of solution of (A.1.33)

Solution of (A.1.33) by reduction to a set of algebraic equations.

We proceed to solve (A.1.33) on the surface of the sphere.

In Chap. I it was suggested that equations similar to (A.1.33) could be solved by assuming a piecewise linear variation of the unknown function with time, assuming that the function was zero at $t=0+$. This is not the case here, however, by virtue of the right hand side of (A.1.33) which is different from zero at $t=0+$. On the other hand it is convenient to solve an auxiliary equation in which the right hand side is $\frac{2}{3}E[\xi(r_2,t)]$. i.e.

$$\psi(t) + \frac{1}{3K}\int_{0}^{t} E[\xi(r_2,t)-\xi(r_2,\tau)]\frac{\partial\psi}{\partial\tau}\,d\tau = \frac{2}{3}E[\xi(r_2,t)] \qquad \text{(A.1.35)}$$

On the surface of the sphere we have from the example of Ref. $\xi = 10^{3.6}\,t$, and

$$E[\xi(r_2,t)-\xi(r_2,\tau)] = E[10^{3.6}(t-\tau)] = \tilde{E}(t-\tau) \qquad \text{(A.1.36)}$$

If we look at (A.1.35) in the (r, ξ) plane, then, after Laplane transform and in view of (A.1.35), and (A.1.33) we get

$$S_{r_2} = \int_0^t \psi(t-\tau)\,\frac{\partial h}{\partial \tau}\,d\tau \tag{A.1.37}$$

after comparison of (A.1.37) with (A.1.30) it transpires that

$$\psi(t) = R\left[\xi(r_2, t)\right] \tag{A.1.38}$$

Since R is known, an estimate of the accuracy of the method can be made on the basis of solution of Eq. (A.1.35). Again, of course, $\psi(0+) \neq 0$.

Hence following Ref. [12] :

$$\psi(t) = H(t)\,\psi(0+) + \psi^*(t) \tag{A.1.39}$$

where H(t) is the unit step function, and .

Substituting (A.1.39) in (A.1.35) and in view of (A.1.36):

$$\psi(0+)\,H(t) + \psi^*(t) + \frac{\psi(0+)}{3K}\,\tilde{E}(t) + \frac{1}{3K}\int_{0+}^{t} \tilde{E}(t-\tau)\frac{\partial \psi^*}{\partial \tau}\,d\tau = \frac{2}{3}\,\tilde{E}(t) \tag{A.1.40}$$

Putting $t = 0+$ we find

$$\psi(0+) = \frac{\tilde{E}(0+)}{1 + \dfrac{\tilde{E}(0+)}{3K}} \tag{A.1.41}$$

In view of (A.1.41), (A.1.40) becomes

$$\psi^{*}(t) + \frac{1}{3K}\int_{0+}^{t} \tilde{E}(t-\tau)\frac{\partial\psi^{*}}{\partial\tau}\,d\tau = -\psi(0+)\left\{1 - \frac{\tilde{E}(t)}{\tilde{E}(0+)}\right\} \qquad \text{(A.1.42)}$$

Eq. (A.1.39) is a necessary step and the omission of the initial conditions would introduce significant error in the solution of (A.1.35).

The right hand side of (A.1.42) is now a known function which we denote by g(t) i.e.

$$g(t) = -\psi(0+)\left\{1 - \frac{\tilde{E}(t)}{\tilde{E}(0+)}\right\} \qquad \text{(A.1.43)}$$

and (A.1.42) becomes

$$\psi^{*}(t) + \frac{1}{3K}\int_{0}^{t} \tilde{E}(t-\tau)\frac{\partial\psi^{*}}{\partial\tau}\,d\tau = g(t) \qquad \text{(A.1.44)}$$

Because of the nature of $\tilde{E}(t)$ Fig. (A.1.8) g(t) has a very high gradient at t = o.

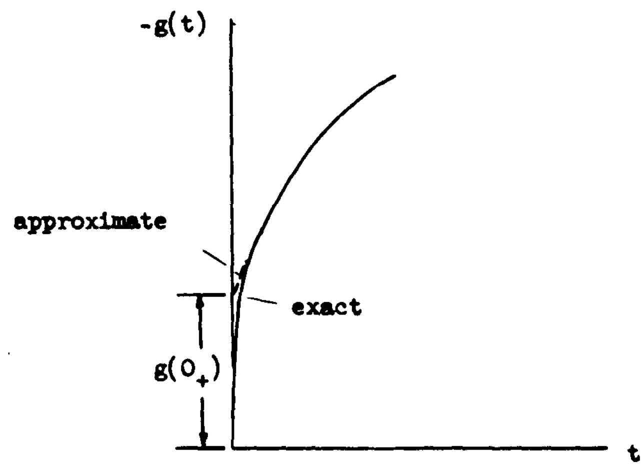


Figure (A.1.5)

This is a potential source of error for a finite initial interval, unless the first intervals are taken extremely small. To avoid such a lengthly numerical computation we approximate here the high gradient of g(t) at t = o_+ by a finite step at t = o as shown in Fig. (A.1.5).

Thus again we write

$$g(t) = g(o{+}{+})\, H(t) + g'(t) \qquad \text{(A.1.45)}$$

and consequently we can write:

$$\psi^*(t) = \psi^*(o{+}{+})\, H(t) + \psi'(t) \qquad \text{(A.1.46)}$$

Substituting (A.1.46) and (A.1.44) we obtain

$$\psi^*(o{+}{+}) = \frac{g(o+)}{1 + \dfrac{\tilde{E}(o+)}{3K}} \qquad \text{(A.1.47)}$$

and

$$\psi'(t) + \frac{1}{3K}\int_o^t \tilde{E}(t-\tau)\frac{\partial \psi'}{\partial \tau}\, d\tau = g(t) - \psi^*(o{+}{+})\left(1 + \frac{\tilde{E}(t)}{3K}\right) \qquad \text{(A.1.48)}$$

The right hand side of (A.1.48) is now a well behaved function and (A.1.48) can be solved by piecewise linearization of   over small finite intervals.

Naturally:

$$\psi(t) = \psi(o+)\, H(t) + \psi^*(o{+}{+})\, H(t - o{+}{+}) + \psi'(t) \qquad \text{(A.1.49)}$$

Solution of $\psi(t)$ has been obtained over a total time of .20 hours, in the following intervals:

o, o+, o++, .01, .02, .04, .08, .12, .20

$\psi(t)$, which is identical to R $\left[\xi (r_2, t) \right]$, thus obtained is compared against the exact solution of Ref.[7] . The agreement between the two solutions, shown in Fig. (A.1.10) is exceedingly good.

The hoop stress σ_θ was then calculated from (A.1.37) and (A.1.26) and is compared with exact σ_θ from (A.1.34) in Fig. (A.1.11) Again agreement is very satisfactory.

Solution of (A.1.33) by an iteration technique.

Since $\psi(o+)$ has been found, eq. (A.1.41), we solve (A.1.44) instead. In other words in the zeroth approximation K is considered infinite for all values of time except at t = o+. This apparently artificial physical consequence can be appreciated if one looks at $\nu(t)$.

From (1.54)

$$\nu(t) = \frac{1}{2} - \frac{E(t)}{3K} \tag{A.1.50}$$

Note that $\nu(t) \rightarrow 1/2$ as $t \rightarrow \infty$, See Fig. (A.1.6)

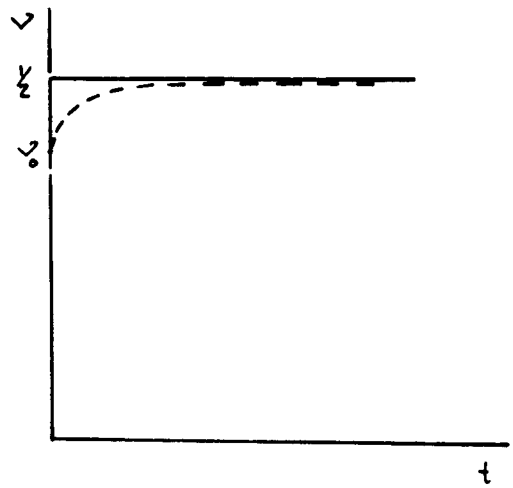


Figure (A.1.6)

The above values of K imply that $\nu = \nu_o$ at t = o but $\nu = \frac{1}{2}$ for $t > o$, in other words we have approximated $\nu(t)$ by the thick line shown in Fig. (A.1.6).

$\psi^*(t)$ is then found from the recurrence relation

$$\psi_n^* + \int_0^t \tilde{E}(t-\tau)\frac{\partial \psi_{n-1}^*}{\partial \tau} d\tau = g(t) \qquad (A.1.51)$$

for n = 1, 2 ... where

$$\psi_o^* = g(t) \qquad (A.1.52)$$

Values of $\psi(t)$ obtained by this method are also shown in Fig. (A.1.10), and these too compare favorably with the exact solution. The hoop stress σ_θ is also found and plotted in Fig. (A. 1.11), with good agreement.

Stresses in Viscoelastic Solids due to Cyclic Temperature Histories

The effects of cyclic temperatures on the stresses in viscoelastic bodies are not well understood, and this is not surprising in view of the analytical difficulties inherent in the investigation of this problem. The critical effect of temperature on material properties, renders the problem non-linear in so far as the principle of superposition no longer holds with regard to separate temperature histories. That is, if $\overset{(A)}{T}(x_K, t)$ produces $\overset{(A)}{\sigma_{ij}}(x_K, t)$ and $\overset{B}{T}(x_K, t)$ produces $\overset{(B)}{\sigma_{ij}}(x_K, t)$ then:

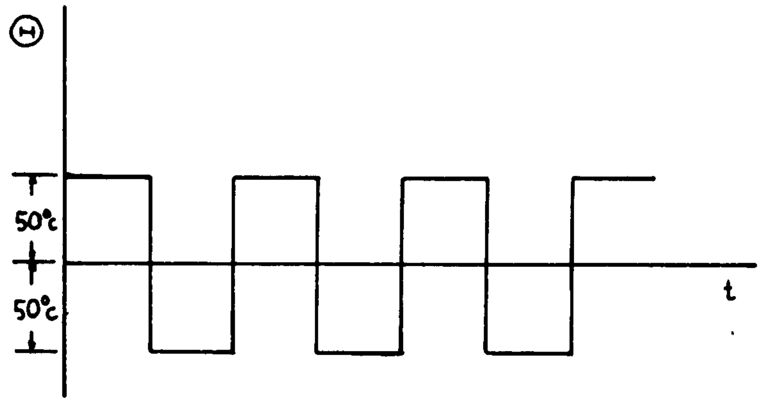
$$\overset{(A)}{T}(x_K, t) + \overset{(B)}{T}(x_K, t)$$

does not give rise to

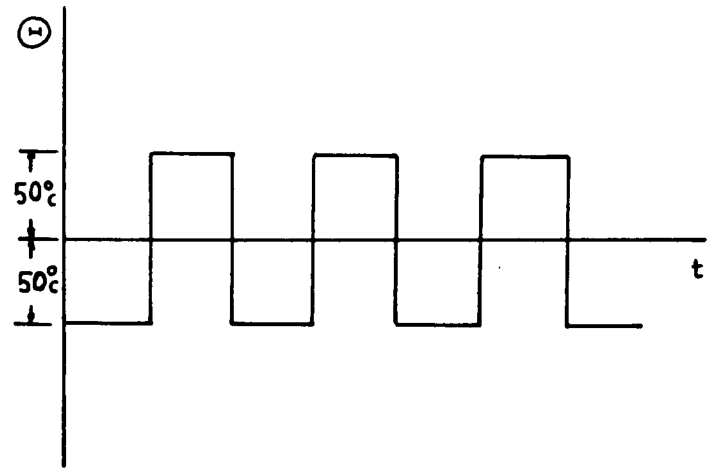
$$\overset{(A)}{\sigma_{ij}}(x_K, t) + \overset{(B)}{\sigma_{ij}}(x_K, t)$$

The superposition principle will hold only in the particular case where $\overset{(A)}{T}$, $\overset{(B)}{T}$ are simultaneously taken to affect to material properties but their effects on the strains are considered separately. In such a context, however, superposition will be of little value.

To investigate the phenomenon at all, we take the simple case of the viscoelastic slab, and subject its surface to cyclic temperature changes shown in Fig. (A.1.7).



Cycle (a)



Cycle (b)

Figure (A.1.7)

We examine the effect of the two cycles shown, on the stress at the surface of the infinite slab, when $\Theta_0 = 50^\circ C$, and the datum temperature is $70^\circ C$.

Moreover we assume that the slab is made of LPC-543A propellant for which the shift factor is a well known function of temperature, see Ref. , p. 55.

For this particular material it is found that the shift factor is 75 when $\Theta = 50^\circ C$, and $\frac{1}{75}$ when $\Theta = -50^\circ C$ or

$$\xi = \begin{cases} 75t & \Theta = 50^\circ C \\ \frac{1}{75}t & \Theta = -50C \end{cases} \tag{A.1.53}$$

In the notation of Ref. [7], on the surface of the slab

$$\hat{\sigma}_x(a, \xi) = -3\alpha_0 \int_0^\xi R(\xi - \xi') \frac{\partial \Theta}{\partial \xi'} d\xi' \tag{A.1.54}$$

Fig's (A.1.12) and (A.1.13) show ξ as a function of t and Θ as a function of ξ. It is easily seen that Θ can be expressed in the following form.

$$\Theta = \Theta_0 \left\{ H(\xi) - 2\,H(\xi - \xi_1) + 2\,H(\xi - \xi_2) - \cdots \right\} \tag{A.1.55}$$

and consequently

$$\frac{\partial \Theta}{\partial \xi} = \Theta_0 \left\{ \delta(\xi) - 2\delta(\xi - \xi_1) + 2\delta(\xi - \xi_2) - \cdots \right\} \tag{A.1.56}$$

where $\delta(\xi-\xi_r)$ is the delta function at $\xi = \xi_r$. Substitute (A.1.56) in (A.1.54) and integrating we get

$$-\frac{1}{3\alpha_0\Theta_0}\sigma_x\{a,\xi(t)\} = R(\xi) - 2R(\xi-\xi_1) + 2R(\xi-\xi_2)\ldots \quad \text{(A.1.57)}$$

From (A.1.57) we can easily find σ_x as a function of time for two temperature cycles, see Figs. (A.1.14) and (A.1.15).

It can be seen that in both cases σ_x increases asymptotically to values which are almost twice the maximum σ_x for identical temperature cycles applied to an elastic slab. This phenomenon will take place irrespective of the period of the cyclic variation, and is due to the difference in the relaxation rates at the lower and higher temperatures.

Though this result is important it cannot be readily generalized either for more complex temperature histories or for more intricate boundary value problems. It does however bring out a phenomenon that must be considered in the design of solid propellant configurations.

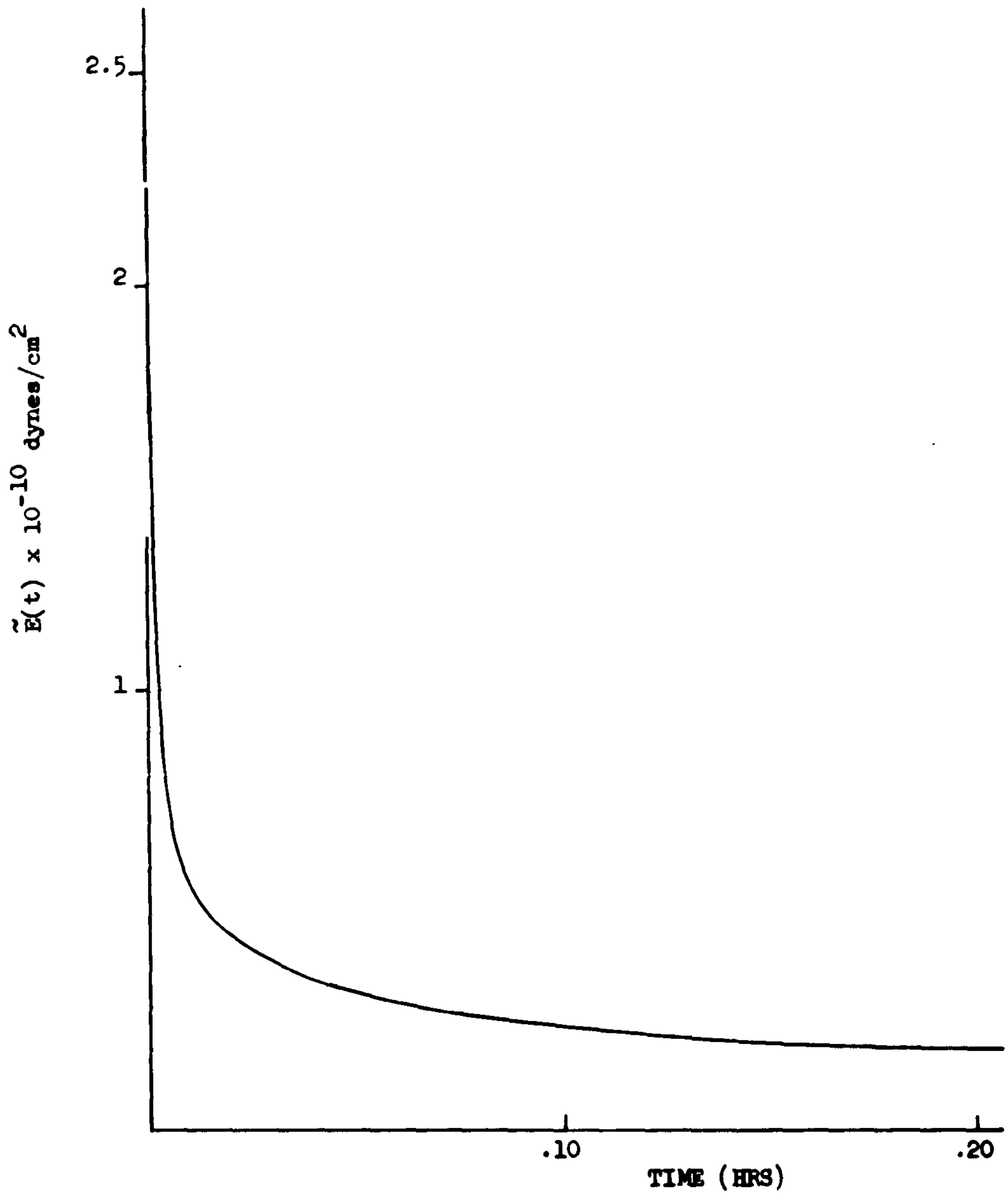


Figure (A.1.8)

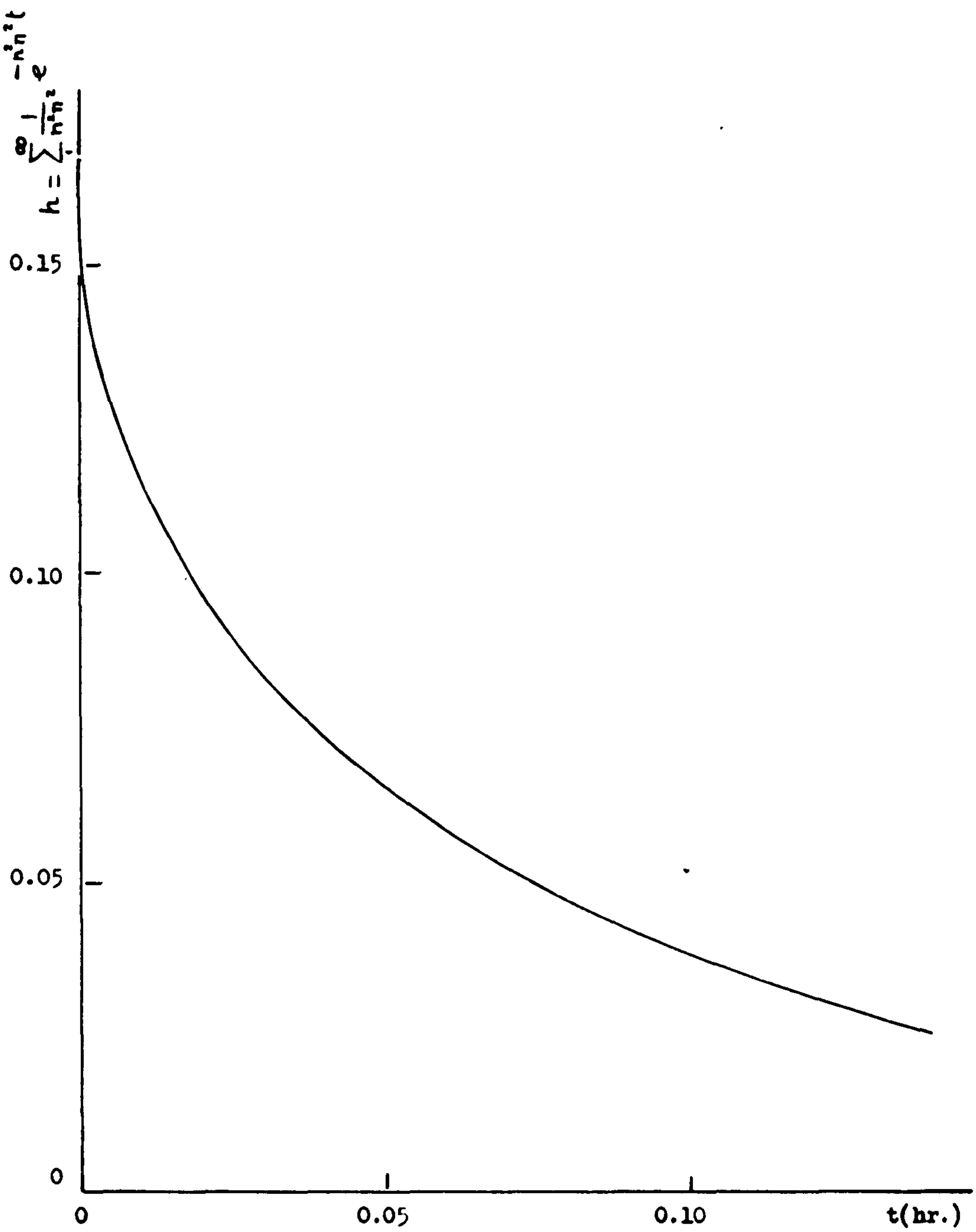


Figure (A.1.9)

FUNCTION h(t) vs. TIME

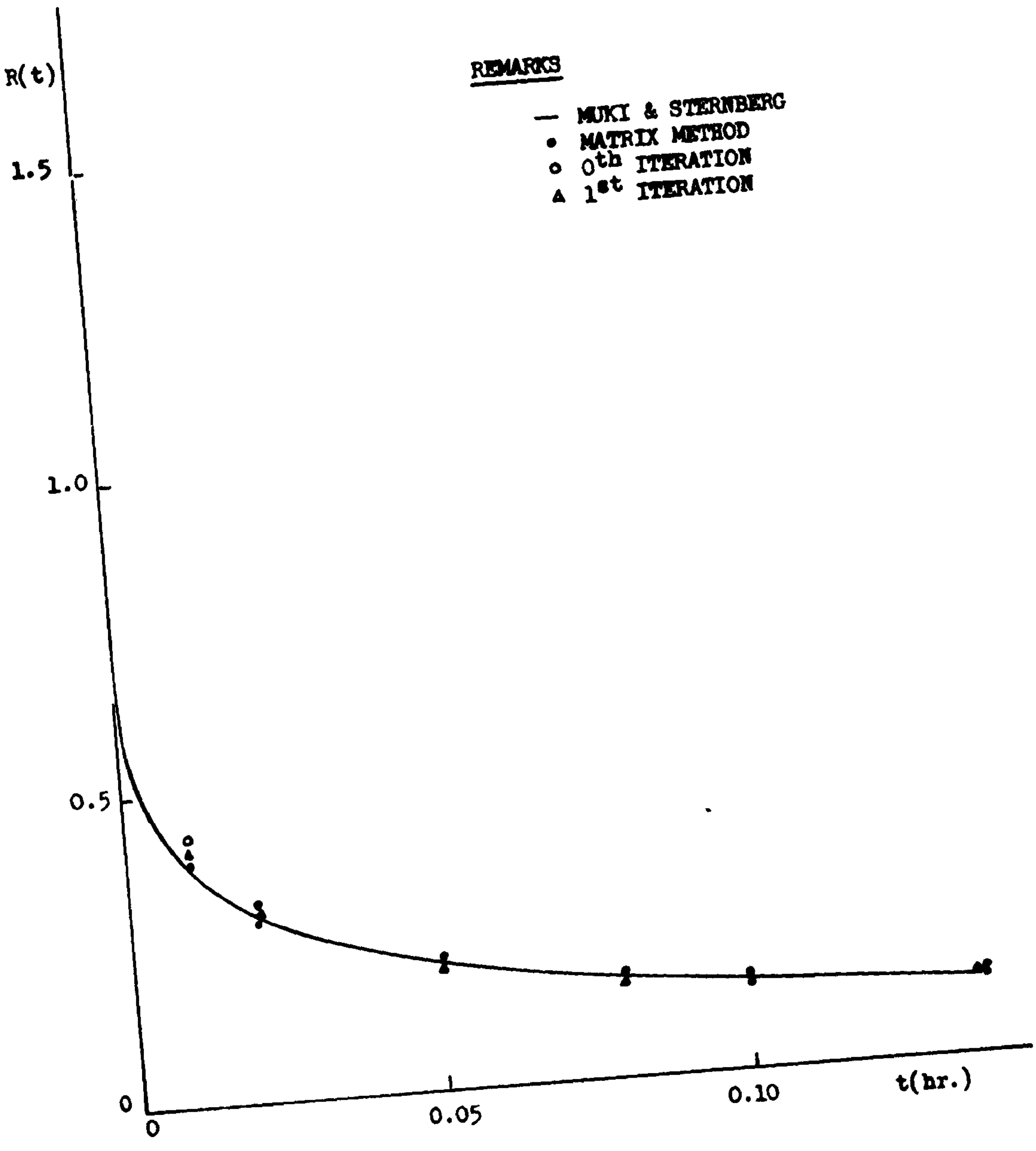


Figure (A.1.10)
SPHERE R(t) vs. t AT SURFACE

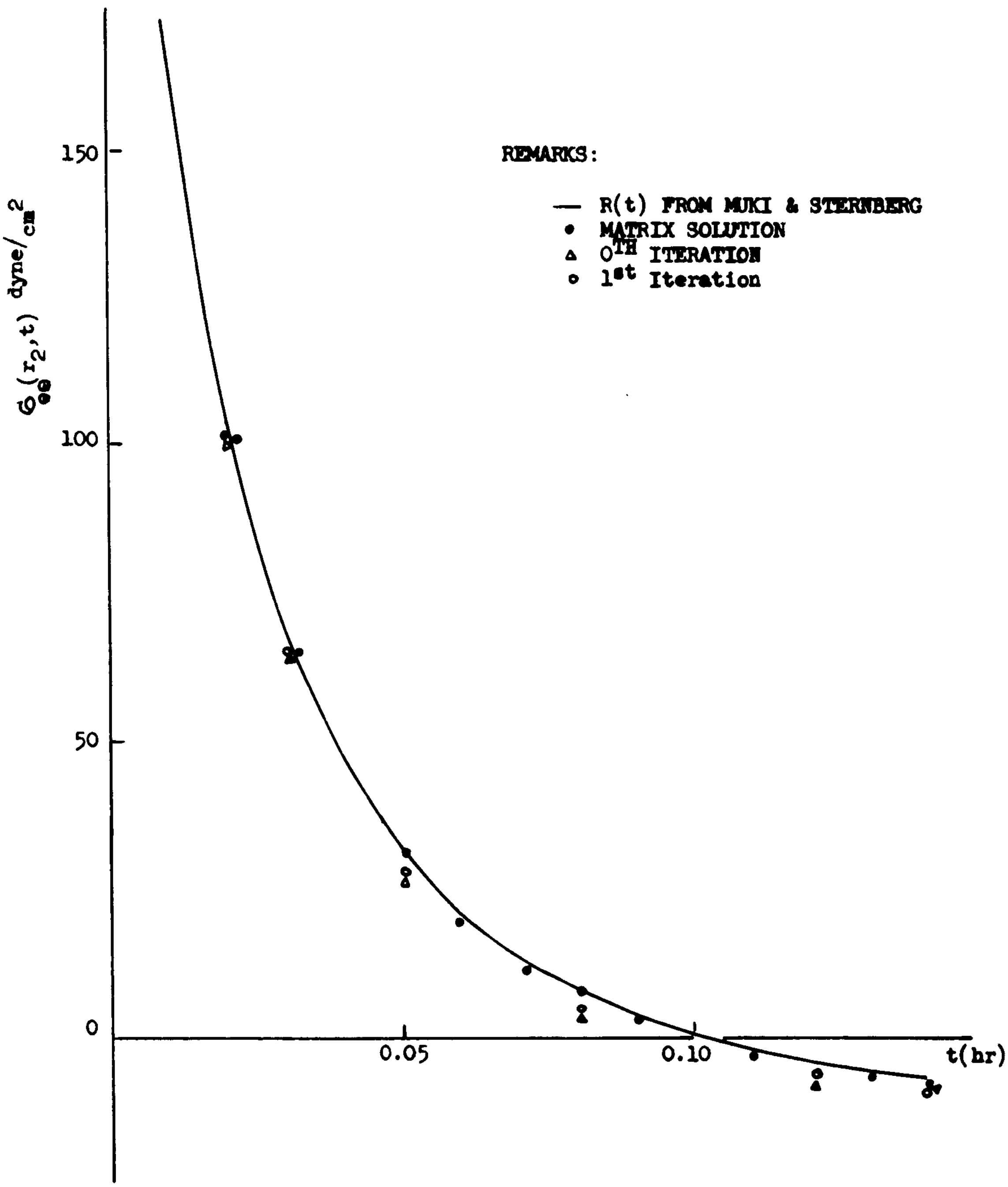


Figure (A.1.11)

SPHERE TIME DEPENDENCE OF $\sigma_{\theta\theta}$ AT SURFACE

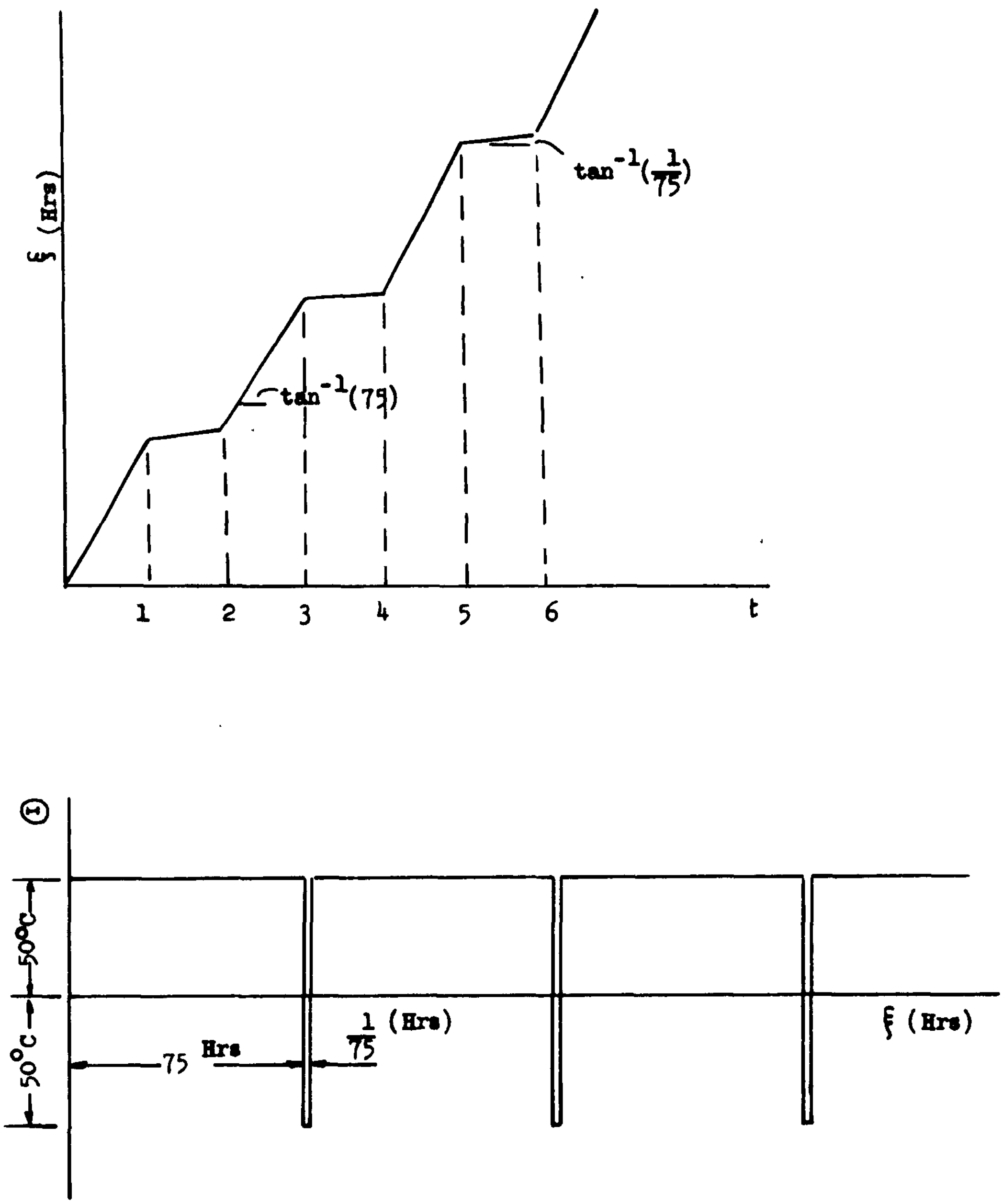
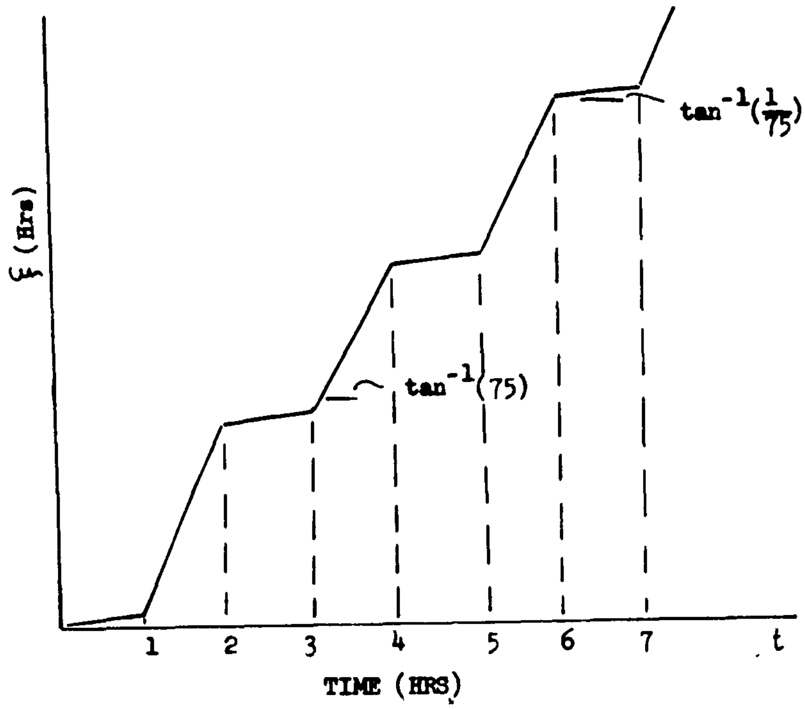


Figure (A.1.12)

Cycle (a)



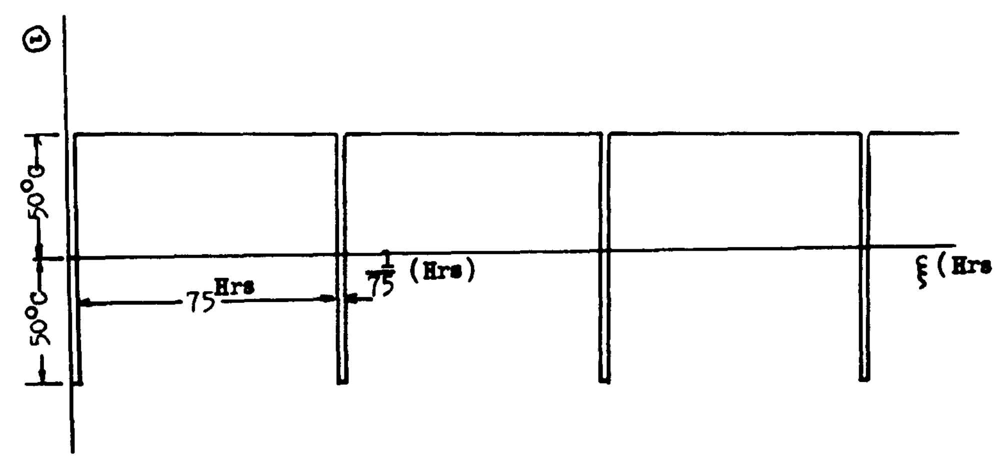
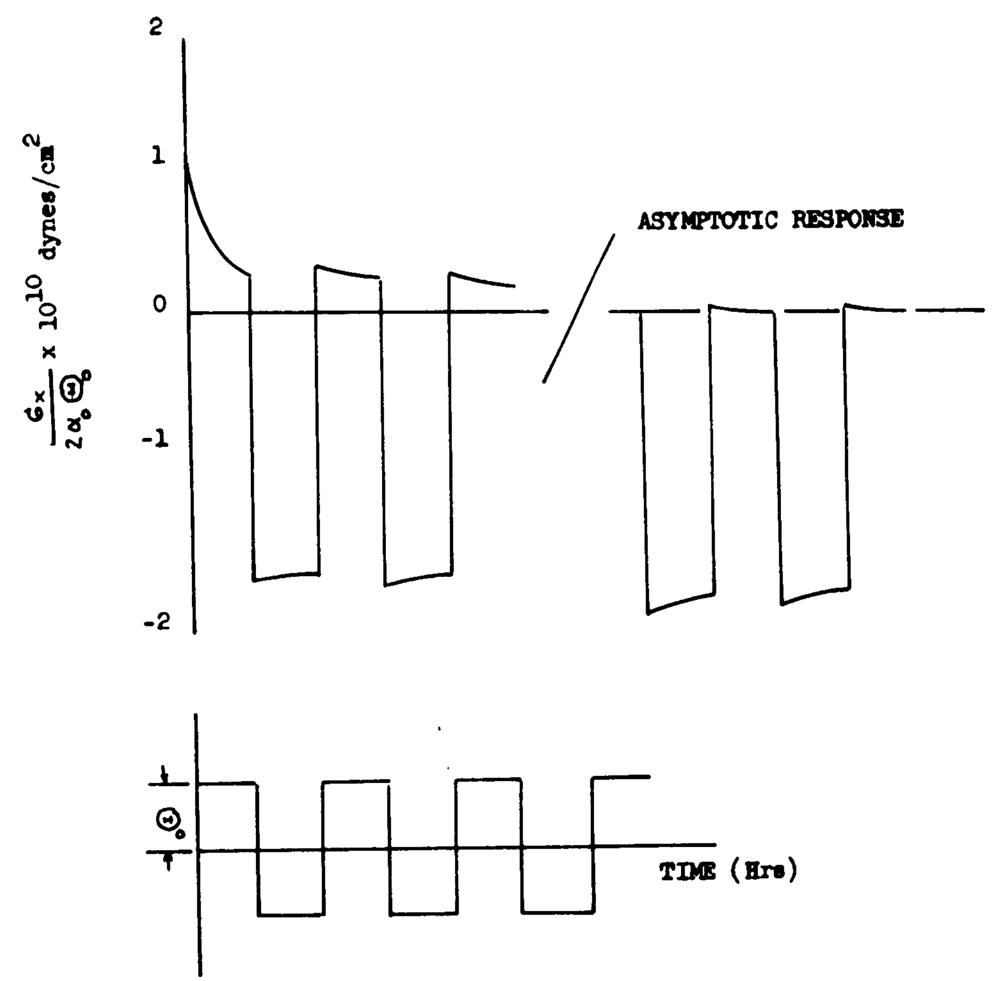


Figure (A.1.13)
Cycle (b)



Θ_o = 50°C, TEMPERATURE HISTORY

Figure (A.1.14)

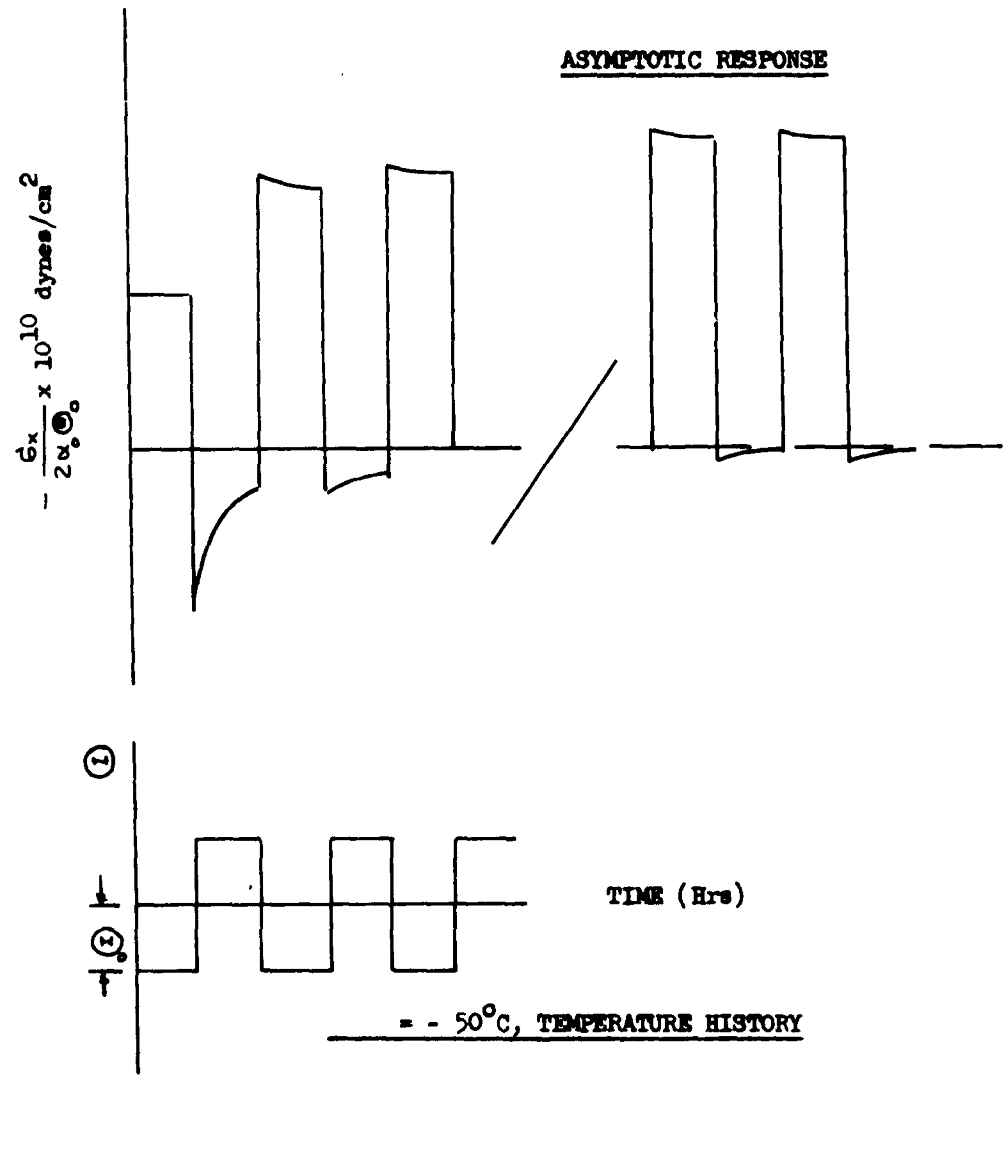

ASYMPTOTIC RESPONSE
$-\frac{\sigma_x}{2\alpha_o \Theta_o} \times 10^{10}$ dynes/cm^2
Θ
Θ_o
TIME (Hrs)
= - 50°C, TEMPERATURE HISTORY

Figure (A.1.15)

APPENDIX II

Error Analysis of Approximate Solutions Developed in the First Chapter

Introduction

In Chapter I it was shown that the solution of viscoelastic boundary value problems associated with thermorheologically simple viscoelastic solids in non-uniform transient temperature fields reduces to the solution of a Volterra integral equation of the second kind. In some cases [7] the integral equation can be expressed in a convolution form in the reduced variable and can then be solved by taking Laplace transform.

Where this, however, is impossible or where the relaxation moduli of the material are in the form of experimental curves, which is invariably the case, Volterra integral equations can be solved approximately by reduction to a set of algebraic equations as was shown in Chapter I. The solution of these equations is easy because the matrix of the coefficients is triangular.

In this report we investigate the error inherent in this procedure by finding first an auxiliary solution. Upper and lower bounds to this solution are then established, and these are utilized to establish an estimate to the aforementioned error.

Upper and Lower Bounds to Solutions of Volterra Integral Equations

Consider the integral equation

$$\phi(t) + \lambda \int_0^t K(t,\tau) \frac{d\Phi}{d\tau} d\tau = f(t) \qquad \text{(A.2.1)}$$

where

$$K(t,\tau) = \int_a^b R(x)\, G\{\xi(x,t) - \xi(x,\tau)\}\, dx \qquad \text{(A.2.2)}$$

and λ is a positive real number, and $R(x)$ is an integrable non negative function.

The symbol ξ denotes the reduced time variable given by the relation

$$\xi = \int_0^t a(x,t)\, dt \qquad \text{(A.2.3)}$$

and

$$a(x,t) = e^{f\{T(x,t)\}} \qquad \text{(A.2.4)}$$

where $f(T)$ is the shift function.

Whatever the form of $T(x,t)$, for any x, $a(x,t)$ is a non-negative function of t and hence $\xi(x,t)$ is a monotonically increasing function of t.

Eq. (A.2.2) may be written in the form

$$K(t,\tau) = \sum_n R_n G\{\xi(x_n,t) - \xi(x_n,\tau)\}\Delta x_n \qquad \text{(A.2.5)}$$

R_n and x_n assuming appropriate values.

Examining a typical term of (A.2.5) we deduce the following properties for $K(t,\tau)$

$$K(t,t) = K_o \qquad \text{(A.2.6)}$$

For any t, $K(t,\tau)$ is a monotonically increasing function of τ, since $G(t)$ is a monotonically decreasing function of t, this being the property of the relaxation modulus of linear viscoelastic solids. Also

$$K(t,o) = K(t) \qquad \text{(A.2.7)}$$

where $K(t)$ is monotonically decreasing in t. Fig. (A.2.1

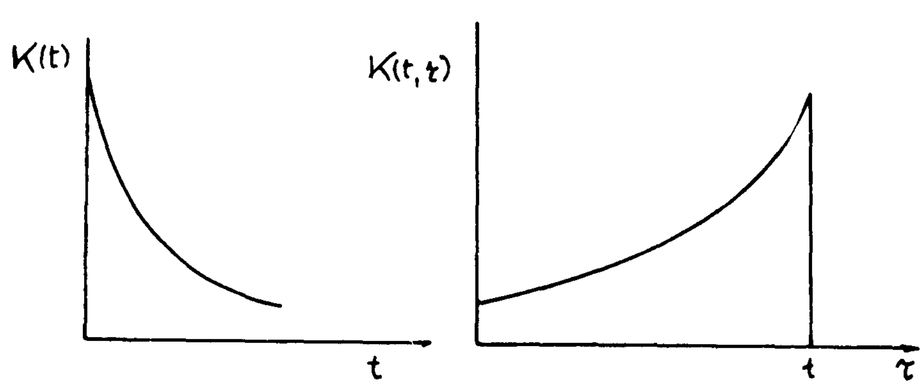


Figure (A.2.1)

Also

$$f(t) = H(t)\,f(t) \qquad \text{(A.2.8)}$$

where $H(t)$ is the Heaviside unit function. Without loss of generality we also assume that $f(0)=0$, hence $\phi(0)=0$, because the case of $f(0)\neq 0$ can be reduced to the previous case as will be shown later.

Eq. (1) can be reduced to the standard form of the Volterra integral equation of the second kind by integration by parts, i.e.

$$\phi(t) + \lambda K(t,\tau)\,\phi(\tau)\Big|_0^t - \lambda\int_0^t \frac{\partial K(t,\tau)}{\partial\tau}\phi(\tau)\,d\tau = f(t) \qquad \text{(A.2.9)}$$

or

$$\phi(t) - \lambda'\int_0^t K^*(t,\tau)\,\phi(\tau)\,d\tau = f'(t) \qquad \text{(A.2.10)}$$

where

$$K^*(t,\tau) = \frac{\partial K(t,\tau)}{\partial\tau} \qquad \text{(A.2.11)}$$

$$\lambda' = \frac{\lambda}{1+\lambda K_0}\ , \quad f'(t) = \frac{f(t)}{1+\lambda K_0} \qquad \text{(A.2.12)}$$

From (4.2.5)

$$\frac{\partial K(t,\tau)}{\partial \tau} = -\sum_{n} R_{n}\, a(\tau)\, \dot{G}\left\{ \xi(x_n,t) - \xi(x_n,\tau) \right\} \qquad \text{(A.2.13)}$$

Now since $a(\tau)$ is always positive and $G(t)$ is monotonically decreasing, $\overset{\circ}{G}$ is non-positive and hence: $\overset{*}{K}(t,\tau)$ is non-negative.

<u>**Lemma.**</u> Subject to the above restrictions on $K(t,\tau)$, $\Phi(t)$ is non-negative and non-decreasing if $f(t)$ is non-negative and non-decreasing.

<u>Proof:</u>

Eq. (A.2.10) is the standard form of the Volterra integral equation of the second kind.

From the theory of integral equations it is proved [11] that

$$\phi(t) = f'(t) + \int_0^t \left\{ \sum_{m=1}^{\infty} \gamma^m \overset{*}{K}_m(t,\tau) \right\} f'(\tau)\, d\tau \qquad \text{(A.2.14)}$$

where $\overset{*}{K}_m(t,\tau)$ is defined by the recurrence relation

$$\overset{*}{K}_{m+1}(t,\tau) = \int_0^t \overset{*}{K}(t,\eta)\, \overset{*}{K}_m(\eta,\tau)\, d\eta \qquad \text{(A.2.15)}$$

and

$$\overset{*}{K}_1(t,\tau) = \overset{*}{K}(t,\tau) \qquad \text{(A.2.16)}$$

K_m^* is known as the mth iterated Kernel. Since $K_m^*(t,\tau)$ is non-negative all the iterated Kernels are non-decreasing in τ. Also since λ' is a positive member it follows that if $f'(t)$ is a non-decreasing function $\phi(t)$ is also non-decreasting.

We shall utilize this important fact in what follows. In general $f(t)$ will be an arbitrary function so we seek the solution for $\phi(t)$ when $f'(t)$ is a simple non-decreasing function such as $H(t)$ and use the solution to construct another solution for arbitrary $f'(t)$.

Reformulation of the Problem

Given (A.2.1) where $f(t)$ is an arbitrary function we seek the solution to the equation

$$\psi(t) + \lambda \int_0^t K(t,\tau)\frac{\partial \psi}{\partial \tau}\, d\tau = H(t) \tag{A.2.17}$$

then

$$\phi(t) = \int_0^t \psi(t-\tau)\frac{\partial f}{\partial \tau}\, d\tau \tag{A.2.18}$$

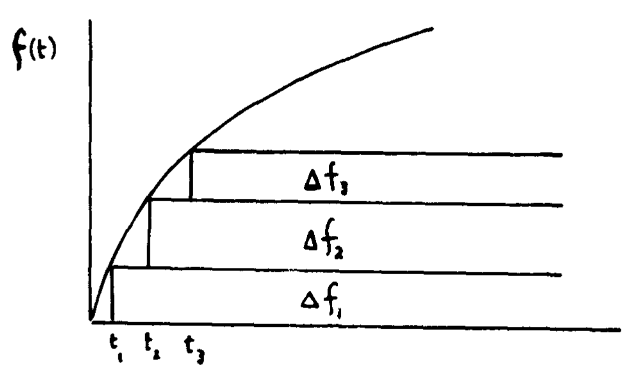


Figure (A.2.2)

Proof:

$$f(t) = \Delta f_1 H(t-t_1) + \Delta f_2 H(t-t_2) + \cdots + \Delta f_n(t-t_n) \quad \text{(A.2.19a)}$$

See Fig. (A.2.2).

In view of (A.2.18)

$$\phi(t) = \Delta f_1 \psi(t-t_1) + \Delta f_2 \psi(t-t_2) + \cdots + \Delta f_n \psi(t-t_n) \quad \text{(A.2.19b)}$$

In the limit

$$\phi(t) = \int_0^t \psi(t-\tau) \frac{df}{d\tau} d\tau \qquad \text{(A.2.19c)}$$

However $\psi(0+) \neq 0$ so we reduce the solution to the standard case by writing

$$\psi(t) = \psi_0 H(t) + \psi^*(t) \qquad \text{(A.2.20)}$$

where $\quad \psi^*(0) = 0$

Then substituting for $\psi(t)$ in (A.2.18) we get

$$\psi_0 H(t) + \psi^*(t) + \lambda K(t) \psi_0 + \lambda \int_0^t K(t,\tau) \frac{d\psi^*}{d\tau} d\tau = H(t) \qquad \text{(A.2.21)}$$

Putting $t = 0+$ we get

$$\psi_0 + \lambda K_0 \psi_0 = 1 \tag{A.2.22}$$

hence

$$\psi_0 = \frac{1}{1+\lambda K_0} \tag{A.2.23}$$

and substituting for ψ_0 in (A.2.21) we find

$$\psi^*(t) + \lambda \int_0^t K(t,\tau) \frac{d\psi^*}{d\tau} d\tau = \frac{\lambda}{1+\lambda K_0} \{K_0 - K(t)\} \tag{A.2.24}$$

and the right hand side is now a monotonically increasing function of t .

Let

$$\frac{\lambda}{1+\lambda K_0} \{ K_0 - K(t) \} = g(t) \tag{A.2.25}$$

then (A.2.24) becomes

$$\psi^*(t) + \lambda \int_0^t K(t,\tau) \frac{d\psi^*}{d\tau} d\tau = g(t) \tag{A.2.26}$$

and ψ^* is a monotonically increasing function.

<u>Solution of (A.2.26) by Reduction to an Algebraic Set of Equations</u>

We divide the range of integration into small intervals:

$$(0, t_1)\ (t_1, t_2)\ \ldots\ldots\ldots\ (t_{n-1}, t_n)$$

In the first interval we get

$$\psi^*(t_1) + \lambda \int_0^{t_1} K(t_1, \tau) \frac{d\psi^*}{d\tau} d\tau = g(t_1) \tag{A.2.27}$$

By the mean value theorem

$$\int_0^{t_1} K(t_1, \tau) \frac{d\psi^*}{d\tau} d\tau = K(t_1, \eta) \psi^*(t_1) \tag{A.2.28}$$

where $$0 < \eta \leq t_1 \tag{A.2.29}$$

Letting

$$\psi^*(t_j) \equiv \psi_j^*, \quad g(t_i) \equiv g_j, \quad K(t_i, t_j) \equiv K_{ij} \tag{A.2.30}$$

(A.2.27) becomes

$$\psi_1^* + \lambda K(t_1, \eta) \psi_1^* = g_1 \tag{A.2.31}$$

In view of the properties of $K(t, \tau)$ (A.2.31) may be written in terms of the inequalities

$$\psi_1^* + \lambda K_0 \psi_1^* \geq g_1 \qquad (A.2.32)$$

$$\psi_1^* + \lambda K_{10} \psi_1^* \leq g_1 \qquad (A.2.33)$$

hence:

$$\frac{g_1}{1+\lambda K_0} \leq \psi_1^* \leq \frac{g_1}{1+\lambda K_{10}} \qquad (A.2.34)$$

In the second interval we get

$$\psi_2^* + \lambda K(t_2,\eta_1)\psi_1^* + \lambda K(t_2,\eta_2)(\psi_2^* - \psi_1^*) = g_2 \qquad (A.2.35)$$

where

$$0 \leq \eta_1 \leq t_1 \quad , \quad t_1 \leq \eta_2 \leq t_2 \qquad (A.2.36)$$

We also have the inequalities

$$K(t_2,0) \leq K(t_2,\eta_1) \leq K(t_2,t_1) \qquad (A.2.37)$$

$$K(t_2,t_1) \leq K(t_2,\eta_2) \leq K_0 \qquad (A.2.38)$$

$$\psi_2^* - \psi_1^* > 0 \qquad (A.2.39)$$

In view of inequalities (A.2.37), (A.2.38) and (A.2.39), (A.2.35) becomes:

$$\psi_2^* + \lambda K_{21} \psi_1^* + \lambda K_0 (\psi_2^* - \psi_1^*) \geqslant g_2 \tag{A.2.40}$$

$$\psi_2^* + \lambda K_{20} \psi_1^* + \lambda K_{21} (\psi_2^* - \psi_1^*) \leq g_2 \tag{A.2.41}$$

or

$$\frac{g_2 + \lambda(K_0 - K_{21})\psi_1^*}{1 + \lambda K_0} \leq \psi_2^* \leqslant \frac{g_2 + \lambda(K_{21} - K_{20})\psi_1^*}{1 + \lambda K_{21}} \tag{A.2.42}$$

and similarly for other intervals.

It follows that if we choose

$$\eta_i = t_i \tag{A.2.43}$$

we get a lower bound to ψ^* which we denote by ψ_ℓ^* and by choosing

$$\eta_i = t_{j-1} \tag{A.2.44}$$

we get an upper bound to ψ^* which we denote by ψ_u^* .

Hence ψ^* is bounded by the solutions ψ_ℓ and ψ_u or

$$\psi_\ell \leq \psi \leq \psi_u \tag{A.2.45}$$

Let

$$\phi^{*} = \int_{0}^{t} \psi_{u}(t-\tau)\frac{df}{d\tau}\,d\tau \qquad \text{(A.2.46)}$$

and

$$\phi_{*} = \int_{0}^{t} \psi_{\ell}(t-\tau)\frac{df}{d\tau}\,d\tau \qquad \text{(A.2.47)}$$

Then since $\psi(t-\tau)$ is monotomically decreasing in τ and making use of Bonnet's second mean value theorem

$$\left|\phi^{*}-\phi_{*}\right|_{max} \leq \left|\psi_{u}-\psi_{\ell}\right|_{max}\left|f(\eta')\right|_{max} \qquad \text{(A.2.48)}$$

where $0 \leq \eta' \leq t$

Hence if ε denotes the error in the solution of ϕ for some η_j

$$t_{j-1} \leq \eta_j \leq t_j \qquad \text{(A.2.49)}$$

then

$$\left|\varepsilon\right|_{max} \leq \left|\psi_{u}-\psi_{\ell}\right|_{max}\left|f(\eta')\right|_{max} \qquad \text{(A.2.50)}$$

where again $0 \leq \eta' \leq t$.

Matrix Formulation of the Problem

Consider (Fig. A.2.3) some $\psi(t)$ which is a non-decreasing function of t .

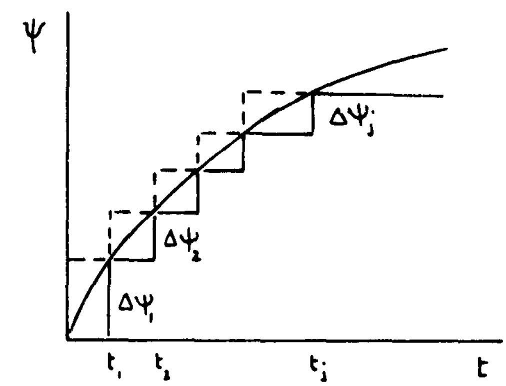


Figure (A.2.3)

is capable of two approximate step-wise representations. One is the full line and the other the dotted line, i.e.

$$\psi_n = \sum_{\nu=0} H(t - t_\nu)\, \Delta \psi_{\nu+1} \tag{A.2.51}$$

or

$$\psi_n = \sum_{\nu=0} H(t - t_{\nu+1})\, \Delta \psi_{\nu+1} \tag{A.2.52}$$

Substitutions in (A.2.27) show that in fact (A.2.51) yields the upper bound solution for ψ and (A.2.52) the lower bound. Thus we get

$$\psi_{u,j} + \sum_{r=0}^{j} K_{jr}\, \Delta\psi_{r+1} = g_j \qquad \text{(A.2.53)}$$

or in matrix form,

$$[H]\{\Delta\psi_u\} + \lambda[K_u]\{\Delta\psi_u\} = \{g\} \qquad \text{(A.2.54)}$$

where

$$[H] = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdots & & 1 \end{bmatrix} \qquad \text{(A.2.55)}$$

and

$$[K_u] = \begin{bmatrix} K_{10} & & & \\ K_{20} & K_{21} & & \\ K_{30} & K_{31} & K_{32} & \\ \cdots & \cdots & \cdots & \\ K_{n0} & \cdots & & K_{n,n-1} \end{bmatrix} \qquad \text{(A.2.56)}$$

From (A.2.54)

$$\{\Delta\psi_u\} = \left[[H] + \lambda[K_u]\right]^{-1}\{g\} \qquad \text{(A.2.57)}$$

and

$$\{\psi_u\} = [H]\left[[H] + \lambda[K_u]\right]^{-1}\{q\} \tag{A.2.58}$$

Similarly

$$\{\psi_\ell\} = [H]\left[[H] + \lambda[K_\ell]\right]^{-1}\{q\} \tag{A.2.59}$$

where

$$[K_\ell] \quad \begin{bmatrix} K_{11} & & & & \\ K_{21} & K_{22} & & & \\ K_{31} & K_{32} & K_{33} & & \\ \cdots & \cdots & \cdots & & \\ K_{n1} & \cdots & & & K_{nn} \end{bmatrix} \tag{A.2.60}$$

Naturally a linear variation in ψ as indicated in Ref. 1 will yield a solution lying between the two bounds.

<u>Example:</u> (From Appendix I)

Consider the slab problem where the relevant integral equation to be solved is

$$\sigma_x + \frac{1}{3K}\int_0^t E(\xi-\xi')\frac{d\sigma_x}{d\tau}d\tau = -2\alpha_0\int_0^t E(\xi-\xi')\frac{d\Theta}{d\tau}d\tau \tag{A.2.61}$$

In this case $\lambda = \frac{1}{3K}$, and

$$K(t,\tau) = E\left(\xi(t) - \xi(\tau)\right) \qquad \text{(A.2.62)}$$

We first solve the auxiliary equation

$$\psi(t) + \frac{1}{3K}\int_0^t E\{\xi(t) - \xi(\tau)\}\frac{d\psi}{d\tau}\,d\tau = H(t) \qquad \text{(A.2.63)}$$

and obtain upper and lower bounds to ψ , by first finding ψ^* as indicated in the previous section.

It is immediately obvious from (A.2.63) that

$$\psi(0+)\left\{1 + \frac{E(0+)}{3K}\right\} = 1 \qquad \text{(A.2.64)}$$

A solution for ψ is obtained in the range $0 \leq t \leq .7$ hours and the range of integration is divided into the following intervals (-oo,o+) (o+,.1) (.1,24) (.24,.4) (.4,.5) (.5,.6) (.6,.7) hours. Under the above conditions we obtain the following values for ψ_l and ψ_u

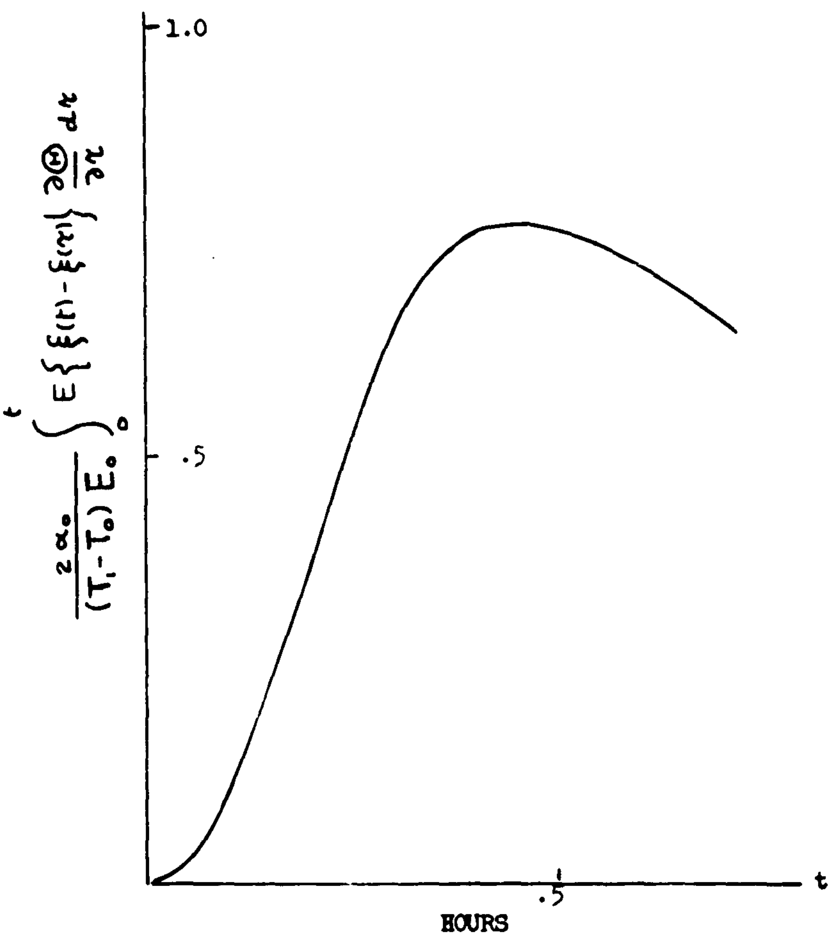
t	ψ_l^*	ψ_u^*	ψ_l	ψ_u
0	0	0	.77321	.77321
.1	.01801	.01845	.79122	.79166
.24	.03591	.03676	.80913	.80997
.4	.05863	.06031	.83184	.83352
.5	.07342	.08160	.85263	.85481
.6	.10629	.10985	.87950	.88306
.7	.12822	.13197	.91043	.90518

For numerical details of the analysis see [12].

From the table above and from Fig. (A.2.5) it can be seen that $\tilde{\psi}$ lies within close bounds the maximum difference between $\tilde{\psi}_u$ and $\tilde{\psi}_l$ being less than 0.5%.

Fig. (A.2.6) shows the exact solution lying between the upper and lower bounds. This is because of the form of $\hat{f}(t)$ in (A.2.1). In general however this may not be the case.

On the other hand $\tilde{\psi}$ helps to bracket the error and guides the choice of the size of the intervals of integration.



1.0
.5
$\frac{2\alpha_0}{(T_1-T_0)E_0}\int_0^t E\{\xi(t)-\xi(\tau)\}\frac{\partial\Theta}{\partial\tau}d\tau$
.5
t
HOURS

Figure (A.2.4)

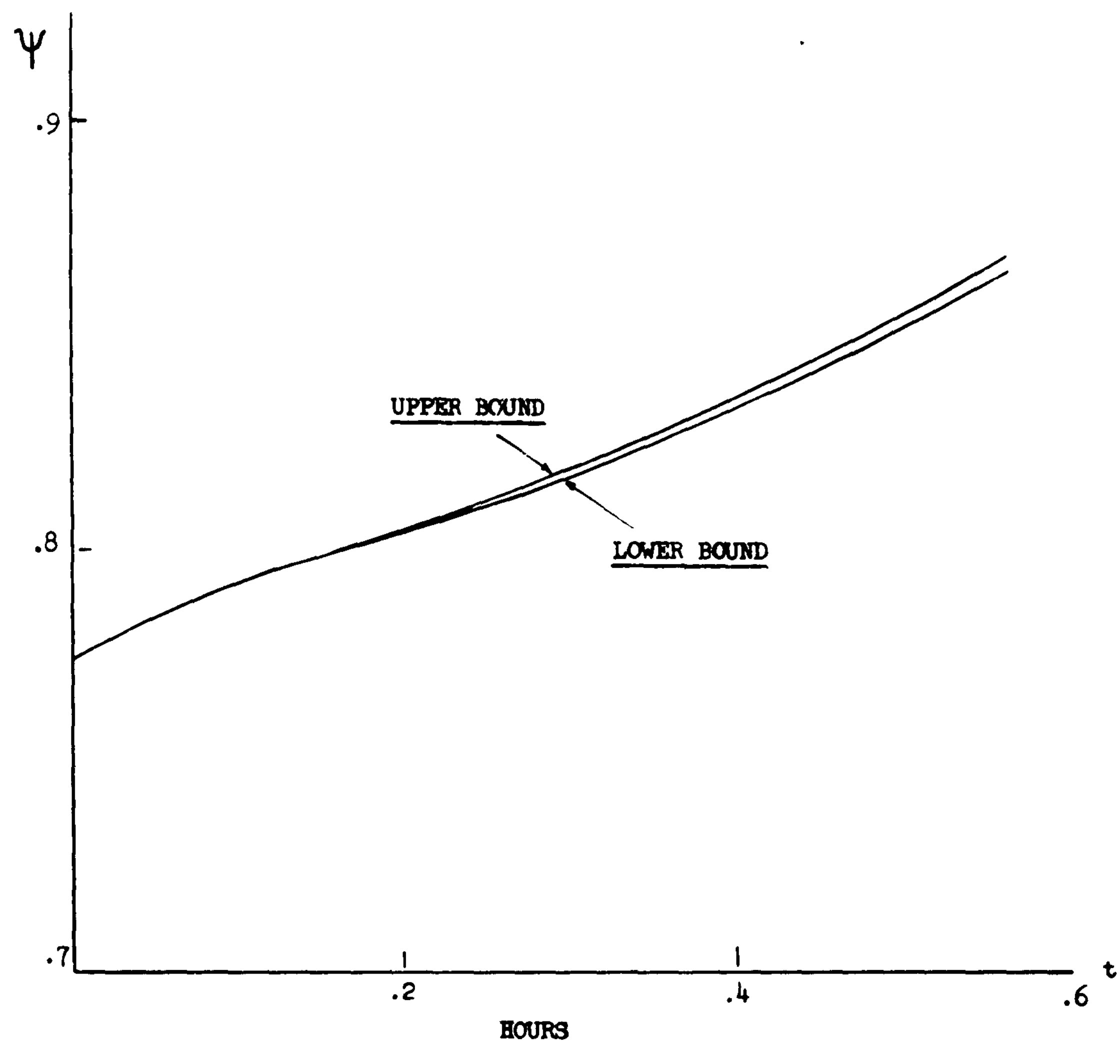


Figure (A.2.5)

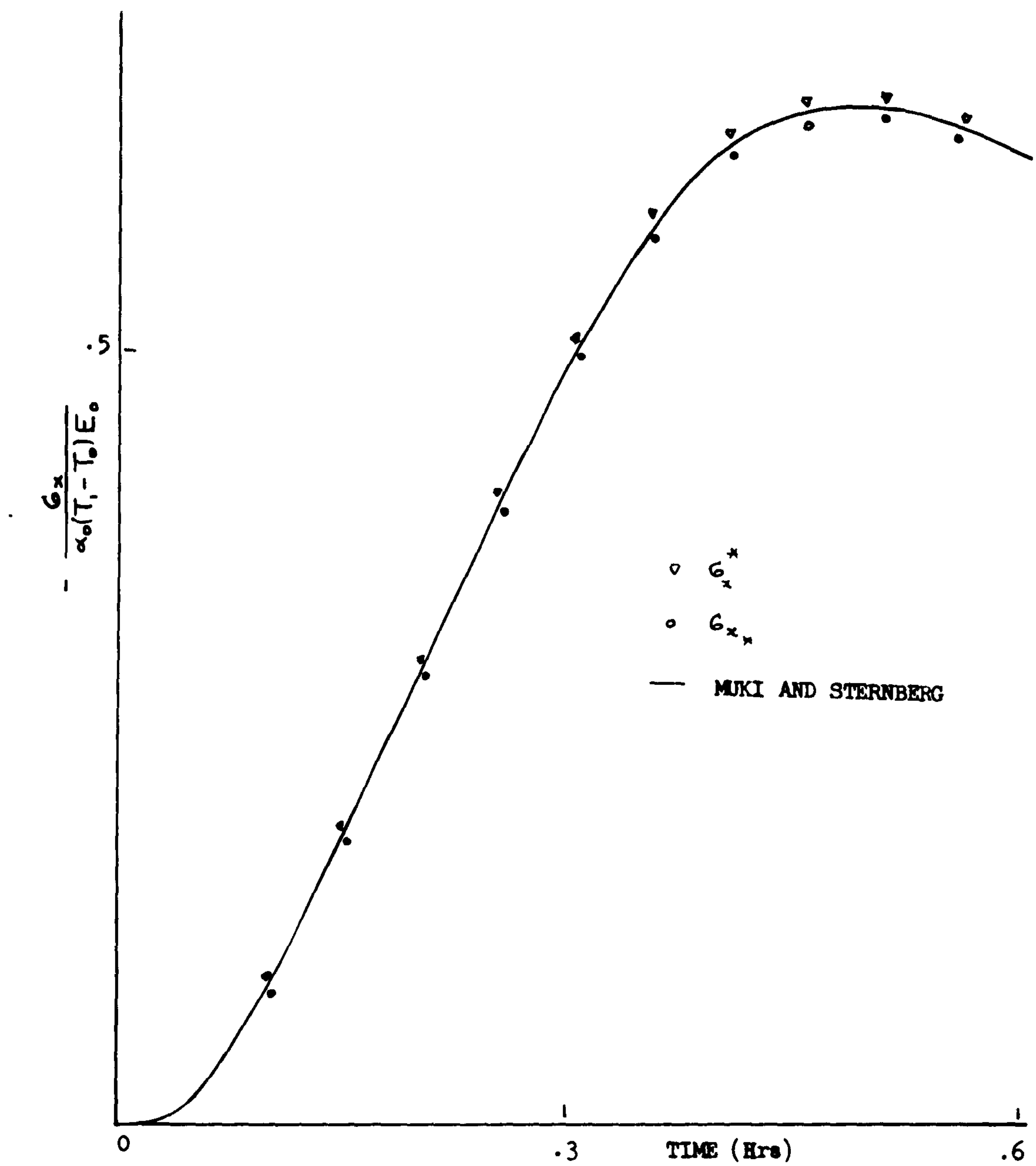

.5
$-\frac{\sigma_x}{\alpha_0(T_1-T_0)E_0}$
σ_x^*
σ_{x_M}
MUKI AND STERNBERG
0
.3
TIME (Hrs)
.6

Figure (A.2.6)

APPENDIX III

Proof of Convergence of the Iteration Solution Obtained in Chapter I.

We finally present a proof of the convergence of the Iteration solution to the cylinder problem, formulated in Chapter I. The solution is obtained by perturbation about the uncompressed state.

Let us expand these stresses and strains into a series of 1/K:

$$\left.\begin{aligned}
\epsilon_\theta &= \epsilon_{\theta_0} + \sum_{n=1}^{\infty} \frac{\epsilon_{\theta n}}{K^n} \\
\epsilon_r &= \epsilon_{r_0} + \sum_{n=1}^{\infty} \frac{\epsilon_{rn}}{K^n} \\
\epsilon &= \epsilon_0 + \sum_{n=1}^{\infty} \frac{\epsilon_n}{K^n} \\
\sigma_r &= \sigma_{r_0} + \sum_{n=1}^{\infty} \frac{\sigma_{rn}}{K^n} \\
\sigma_\theta &= \sigma_{\theta_0} + \sum_{n=1}^{\infty} \frac{\sigma_{\theta n}}{K^n} \\
\sigma &= \sigma_0 + \sum_{n=1}^{\infty} \frac{\sigma_n}{K^n}
\end{aligned}\right\} \tag{A.3.1}$$

where $$\sigma = \sigma_r + \sigma_\theta + \sigma_z \tag{A.3.1a}$$

Since

$$\epsilon = \frac{\sigma}{3K} + 3\alpha_0 \Theta \tag{A.3.2}$$

we obtain

$$\epsilon_0 + \sum_{n=1}^{\infty} \frac{\epsilon_n}{K^n} = \frac{\sigma_0}{3K} + \frac{1}{3}\sum_{n=1}^{\infty} \frac{\sigma_n}{K^{n+1}} + 3\alpha_0 \Theta \tag{A.3.3}$$

Comparing powers of 1/K:

$$\epsilon_n = \frac{1}{3}\sigma_{n-1} \tag{A.3.4}$$

$$\epsilon_0 = 3\alpha_0 \Theta \tag{A.3.5}$$

In general

$$\epsilon = 3\alpha_0 \Theta + \frac{1}{3}\sum_{n=1}^{\infty} \frac{\sigma_{n-1}}{K^n} \tag{A.3.6}$$

Substituting for ϵ in terms of ϵ_θ :

$$\epsilon = \frac{1}{r}\frac{\partial}{\partial r}(r^2 \epsilon_\theta) = 3\alpha_0 \Theta + \frac{1}{3}\sum_{n=1}^{\infty} \frac{\sigma_{n-1}}{K^n} \tag{A.3.7}$$

Integrating (A.3.7):

$$\epsilon_\Theta = \frac{3\alpha_0}{r^2}\int_{r_2}^{r} \rho\,\Theta\,d\rho + \left(\frac{r_2}{r}\right)^2 \epsilon_\Theta(r_2,t) + \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{K^n}\int_{r_2}^{r} \rho\,\sigma_{n-1}\,d\rho \qquad (A.3.8)$$

Substituting for ϵ_Θ from (A.3.1)

$$\epsilon_{\Theta_0} + \sum_{n=1}^{\infty}\frac{\epsilon_{\Theta n}}{K^n} = 3\alpha_0\psi(r,t) + \epsilon_\Theta(r_2,t)\left(\frac{r_2}{r}\right)^2 + \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{K^n}\frac{1}{r^2}\int_{r_2}^{r} \rho\,\sigma_{n-1}\,d\rho \qquad (A.3.9)$$

Comparing coefficients of K,

$$\epsilon_{\Theta_0} = 3\alpha_0\psi(r,t) + \epsilon_\Theta(r_2,t)\left(\frac{r_2}{r}\right)^2 \qquad (A.3.10)$$

$$\epsilon_{\Theta n} = \frac{1}{3}\cdot\frac{1}{r^2}\int_{r_2}^{r} \rho\,\sigma_{n-1}\,d\rho \qquad (A.3.11)$$

Also from (A.3.1),

$$\frac{\partial\sigma_{rn}}{\partial r} = -\,G(r,t) * \left(\frac{\partial\epsilon_\Theta}{\partial r}\right)_n^{*} \qquad (A.3.12)$$

To determine $\frac{\partial\epsilon_\Theta}{\partial r}$ we make use of the expression

---

$$* \quad G * f \equiv \int_0^t G\{\xi(t) - \xi(\tau)\}\frac{\partial f}{\partial \tau}\,d\tau$$

$$\frac{\partial \epsilon_\theta}{\partial r} = \frac{1}{r}\left\{ \epsilon - 2\epsilon_\theta \right\} \tag{A.3.13}$$

which in view of (A.3.11) and (A.3.6) becomes

$$r\frac{\partial \epsilon_\theta}{\partial r} = 3\alpha_0 \Theta + \frac{1}{3}\sum_{n=1}^{\infty}\frac{\sigma_{n-1}}{K^n} - 6\alpha_0 \psi(r_1, t)$$

$$- 2\epsilon_\theta(r_2, t)\left(\frac{r_2}{r}\right)^2 - \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{K^n}\frac{1}{r^2}\int_{r_2}^{r}\rho\,\sigma_{n-1}\,d\rho \tag{A.3.14}$$

In view of (A.3.1) and equating powers of K:

$$\left(\frac{\partial \epsilon_\theta}{\partial r}\right)_0 = \frac{3\alpha_0}{r}\left\{\Theta - 2\psi\right\} - \frac{2}{r}\epsilon_\theta(r_2, t)\left(\frac{r_2}{r}\right)^2 \tag{A.3.15}$$

$$\left(\frac{\partial \epsilon_\theta}{\partial r}\right)_n = \frac{\sigma_{n-1}}{3r} - \frac{2}{3r^3}\int_{r_2}^{r}\rho\,\sigma_{n-1}\,d\rho \tag{A.3.16}$$

Hence from (A.3.12)

$$\sigma_{rn} = \int_{r}^{r_1} G(r, t) * \left(\frac{\partial \epsilon_\theta}{\partial r}\right)_n dr \tag{A.3.17}$$

By definition and in view of (A.3.1),

$$\sigma_n = 3\left\{\sigma_{rn} - S_{rn}\right\}. \tag{A.3.18}$$

where

$$S_{rn} = G(r, t) * e_{rn}(r, t) \tag{A.3.19}$$

It can be readily shown that

$$e_r = \frac{1}{3}\left\{ 2r\frac{\partial \epsilon_\theta}{\partial r} + \epsilon_\theta \right\} \tag{A.3.20}$$

Therefore in view of (A.3.19) and (A.3.20),

$$S_{rn} = \frac{1}{3} G * \left\{ 2r\left(\frac{\partial \epsilon_\theta}{\partial r}\right)_n + \epsilon_{\theta_n} \right\} \tag{A.3.21}$$

and from (A.3.18)

$$\sigma_n = -\left\{\int_{r_1}^{r} G * \left(\frac{\partial \epsilon_\theta}{\partial r}\right)_n dr + \frac{1}{3} G * \left[2r\left(\frac{\partial \epsilon_\theta}{\partial r}\right)_n + \epsilon_{\theta_n}\right]\right. \tag{A.3.22}$$

Eq. (A.3.22) formally completes the cycle of operations in as far as, ϵ_{θ_0} is found from (A.3.10), $\left(\frac{\partial \epsilon_\theta}{\partial r}\right)_0$ from (A.3.15), and σ_{r_0} from (A.3.17). Hence σ_0 is found from (A.3.22), and consequently ϵ_{θ_1} and $\left(\frac{\partial \epsilon_\theta}{\partial r}\right)_1$ can be found from (A.3.11) and (A.3.16) respectively. This procedure can be repeated for higher values of n.

Criteria of Convergence

From (A.3.11), (A.3.16) and (A.3.22) we obtain

$$-\sigma_n = \int_{r_1}^{r} G * \left\{ \frac{\sigma_{n-1}}{r} + \frac{2}{r^3}\int_{r}^{r_2} \rho\,\sigma_{n-1}\,d\rho \right\} dr + \\ + G * \left\{ \frac{2}{3}\sigma_{n-1} + \frac{1}{r^2}\int_{r}^{r_2} \rho\,\sigma_{n-1}\,d\rho \right\} \qquad \text{(A.3.23)}$$

also from (A.3.23)

$$|\sigma_n| \le \left| \int_{r_1}^{r} G * H_{n-1}(r,t)\,dr \right| + \left| G * \left\{ r H_{n-1} + \frac{1}{6}\sigma_{n-1} \right\} \right| \qquad \text{(A.3.24)}$$

where

$$H_{n-1} = \left\{ \frac{\sigma_{n-1}}{r} + \frac{2}{r^3}\int_{r}^{r_2} \rho\,\sigma_{n-1}\,d\rho \right\} \qquad \text{(A.3.24a)}$$

$G(t,\tau,r)$ is a non-negative monotonically increasing function of τ for all r and t .

Hence*

$$\left| \int_0^t G(t,\tau,r)\,\frac{\partial H_{n-1}}{\partial \tau}\,d\tau \right| \le G_0 \left| \hat{H}_{n-1}(r) \right| \qquad \text{(A.3.25)}$$

where

$$\left| \hat{H}_{n-1}(r) \right| = \left| H_{n-1} \right|_{max} \text{ for } \tau = \eta,\ 0 \le \eta \le t .$$

Also

$$\left| \int_{r_1}^{r} G * H_{n-1}\,dr \right| \le \int_{r_1}^{r} \left| G * H_{n-1} \right| dr \le \int_{r_1}^{r} G_0 \left| \hat{H}_{n-1}(r) \right| dr \qquad \text{(A.3.26)}$$

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* For proof see end of Appendix.

in view of (A.3.25).

However

$$\left|\hat{H}_{n-1}(r)\right| \leq \frac{|\hat{G}_{n-1}|}{r} + \frac{2}{r^3}\int_{r}^{r_2} \rho|\hat{G}_{n-1}|\,d\rho \qquad \text{(A.3.28)}$$

Then from (A.3.28), (A.3.25) and (A.3.24)

$$|G_n| \leq G_0\int_{r_1}^{r}\left\{\frac{|\hat{G}_{n-1}|}{r} + \frac{2}{r^3}\int_{r}^{r_2}\rho|\hat{G}_{n-1}|\,d\rho\right\}dr \; +$$

$$G_0\left\{\frac{2}{3}|\hat{G}_{n-1}| + \frac{1}{r^2}\int_{r}^{r_2}\rho|\hat{G}_{n-1}|\,d\rho\right\} \qquad \text{(A.3.29)}$$

We now write

$$\int_{r_1}^{r}\frac{|\hat{G}_{n-1}|}{\rho}\,d\rho = \int_{r_1}^{r}\rho\frac{|\hat{G}_{n-1}|}{\rho^2}\,d\rho \qquad \text{(A.3.30)}$$

Integrating by parts

$$\int_{r_1}^{r}\frac{|\hat{G}_{n-1}|}{\rho}\,d\rho = \frac{1}{r^2}\int_{r_1}^{r}\rho|\hat{G}_{n-1}|\,d\rho + 2\int_{r_1}^{r}\frac{1}{r^3}\left\{\int_{r_1}^{r}\rho|\hat{G}_{n-1}|\,d\rho\right\}dr \qquad \text{(A.3.31)}$$

Introducing (A.3.31) in (A.3.29) we finally obtain

$$\frac{1}{G_0}|G_n| \leq \frac{1}{r^2}\int_{r_1}^{r}\rho|\hat{G}_{n-1}|\,d\rho + \frac{2}{3}|\hat{G}_{n-1}| \qquad \text{(A.3.32)}$$

or

$$\frac{1}{G_o}\left|\sigma_n\right| \leq \left|\sigma_{n-1}\right|_{max}\left\{\frac{1}{\gamma_1^2}\,\frac{\gamma_2^2-\gamma_1^2}{3}+\frac{2}{3}\right\} = \frac{1}{2}\left|\sigma_{n-1}\right|_{max}\left\{\left(\frac{\gamma_2}{\gamma_1}\right)^2+\frac{1}{3}\right\} \qquad (A.3.33)$$

Hence a sufficient (but not necessary) condition of convergence is that

$$\frac{G_o}{K} < \frac{2}{\left(\frac{\gamma_2}{\gamma_1}\right)^2+\frac{1}{3}} \qquad (A.3.34)$$

A typical value of γ_2/γ_1 would be 2.0.

In this case

$$\frac{G_o}{K} < \frac{6}{13} \qquad (A.3.35)$$

which in terms of ν_o becomes,

$$\frac{3(1-2\nu_o)}{2(1+\nu_o)} < \frac{6}{13}$$

or

$$\nu_o > 0.3 \qquad (A.3.36)$$

It is important, however, that smaller values of ν_o do not exclude convergence although their sufficiency cannot be established. Nevertheless we expect that for most viscoelastic materials ν_o will be much nearer 1/2, and hence (A.3.36) is sufficient for most practical purposes.

To Show that if $G(r,t,\tau)$ is Monotonically Increasing in τ

$$\left| \int_0^t G(r,t,\tau) \frac{\partial H}{\partial \tau} d\tau \right| \leq |\hat{H}| \, G_o \qquad (A.3.37)$$

where
$$\hat{H} = H_{max} \quad \text{in} \quad (o, t)$$

$$\left| \int_0^t G(r,t,\tau) \frac{\partial H}{\partial \tau} d\tau \right| = \left| G_1(H_1 - H_o) + G_2(H_2 - H_1) + \cdots + G_n(H_n - H_{n-1}) \right|$$

$$= \left| G_1 H_o + H_1(G_2 - G_1) + \cdots + H_n G_n \right|$$

$$\leq G_1 |H_o| + (G_2 - G_1)|H_1| + \cdots + G_n |H_n| \leq |\hat{H}| \, G_n \qquad (A.3.38)$$

where G_n is some value of G in the nth interval

In the limit $G(r,t)_n \rightarrow G(r,t,t) = G_o$, and therefore

$$\left| \int_0^t G(r,t,\tau) \frac{\partial H}{\partial \tau} d\tau \right| \leq |\hat{H}| \, G_o \qquad (A.3.39)$$

GENERAL DISCUSSION

When the critical dependence of the material properties of visco-elastic solids on temperature was established, it was realized that the prospects of obtaining analytic solutions to the related boundary value problem with arbitrary geometry, were rather poor. In fact, up to the time the present work was undertaken, the only two problems that yielded exact solutions in the formal sense, were those of the infinite state and the sphere with polar symmetry, these solutions being limited to solids with thermorheologically simple behavior.

The purpose of the present work was dual in the sense that though the infinite cylinder was initially the central problem to which a solution was sought, general approximate techniques that would apply to any desired geometry were developed, and tested with very encouraging results in the two cases where exact solutions were known.

The cylinder problem was also treated successfully, and two analytic solutions were given. The perturbution solution can be viewed as an "exact" solution, in as far as the series expansion of the unknown function has been proved convergent, and hence the desired degree of accuracy may be achieved by calculating a sufficient number of terms.

Also, within the limitation of material incompressibility the problems of the sphere and cylinder were examined and solved in the presence of inertia forces. It is worth noting that in the presence of inertia forces, compressibility precludes closed form solutions even in the case of the infinite slab, since the equilibrium equation can no longer be integrated directly.

A perturbation technique is indicated here as well, and will form the basis of future work.

An interesting consequence of the material dependence on temperature, from a structural integrity design standpoint, is the effect of a periodic temperature history on the stresses in a viscoelastic slab.

It was shown in Appendix I that the maximum stress was almost twice the value that would have occurred in an elastic slab under the same temperature history. This is contrary to views expressed in the literature [13] , that stresses due to a step input are likely to be the design stresses.

In fact this phenomenon is only but one facet of the "Thermal Cycling Problem" which is of particular concern in systems with solid propellant configurations. In view of the above result it is natural to inquire into the existence of a critical thermal cycle which will produce the maximum possible stress at some point in the viscoelastic body.

Though the existence of such a cycle is instinctively certain, its determination is a formidable problem, and it requires further studies.

Finally in the third Chapter, the horizontal slump problem of a viscoelastic cylinder under isothermal conditions, contained in an elastic shell is solved formally. Numerical computations are under way.

This is a problem of concern when solid propellant systems have to be stored for a prolonged period of time.

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